THE GENUS SPECTRUM OF CERTAIN CLASSES OF GROUPS

Jay Zimmerman

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May, C.L. and Zimmerman, J., The Symmetric Genus Spectrum of Abelian Groups, (with Coy L. May), submitted.

Definitions

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A Natural Question

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It is not known whether there is a group of symmetric genus n for each value of the integer n.

Upper and Lower Density

Let I be a set of positive integers. For an integer X, let $[1, X]$ be the set of integers between 1 and X and define $I(X) = |I \cap [1, X]|$.

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If I is a set of integers, its *lower* and *upper asymptotic densities*, denoted $\underline{\delta}(I)$ and $\overline{\delta}(I)$, are given by

$$\underline{\delta}(I) = \liminf_{X\to\infty} \frac{I(X)}{X}$$

and

$$\overline{\delta}(I) = \limsup_{X\to\infty} \frac{I(X)}{X}.$$

Density in the Integers

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Now let $\mathcal{S}$ be the set of all positive integers that are the strong symmetric genus of some finite group G. It follows that $\delta(\mathcal{S}) = 1$.

Clearly the 2003 result of May and Zimmerman is considerably stronger than the above density statement.

Density of the Abelian Spectrum

Next let $\mathcal{S}_A$ be the set of all positive integers that are the strong symmetric genus of some finite abelian group G.

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We will show that $\delta(\mathcal{S}_A)$ exists and that it is approximately .3284.

We will also give necessary and sufficient conditions for a positive integer g to be the strong symmetric genus of an abelian group.

Formulas I

Recall that every finite abelian group G has a canonical representation $G \cong Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}$, with standard invariants $m_1, m_2, \ldots, m_r$ subject to $m_1 > 1$ and $m_i | m_{i+1}$ for $1 \leq i < r$.

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Maclachlan (1965) proved that if G is an abelian group of rank $r \geq 3$, with $|G| \geq 10$, then

$$\sigma^0(G) = 1 + \frac{|G|}{2} \min_{0 \leq \gamma \leq r/2} \left\{ 2\gamma - 2 + \sum_{i=1}^{r-2\gamma} \left(1 - \frac{1}{m_i}\right) + \left(1 - \frac{1}{m_{r-2\gamma}}\right) \right\}. \tag{1}$$

Formulas II

For example, when $a > 1$ and $a^3 n \geq 10$, Maclachlan's formula yields

$$\sigma^0(\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}) = 1 - a^2 + a^2(a-1)n.$$

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In particular, when $a = 2$, this reveals that $\mathcal{S}_A$ contains the entire residue class $g \equiv 1 \pmod 4$.

Also when a is odd, $g \equiv 1 - a^2 \pmod{a^2(a-1)}$ and this is equivalent to $g - 1$ is divisible by a^2 for some odd integer a with $(a-1)|g$ by the Chinese Remainder Theorem.

Formulas III

When $b \geq 2$ and $bn > 2$, Maclachlan's formula gives

$$\sigma^0(Z_a \times Z_{ab} \times Z_{abn}) = 1 + b^2 a^2 (a-1) n. \tag{2}$$

In all cases, except when a and b are odd, with $a \equiv 3 \pmod 4$, $g \equiv 1 \pmod 4$.

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Proposition

The spectrum of abelian groups of rank 3 *consists of the congruence class* $g \equiv 1 \pmod 4$ *and the integers* g *satisfying conditions (iii) or (iv) of the Theorem below.*

Main Theorem

Main Theorem

Let $g \geq 2$. Then $g \in \mathcal{S}_A$ if and only if g satisfies one of the following conditions:

(i) $g \equiv 1 \pmod 4$ *or* $g \equiv 55 \pmod{81}$;

(ii) $g - 1$ *is divisible by* p^4 *for some odd prime* p;

(iii) $g - 1$ *is divisible by* a^2 *for some odd integer* a *with* $(a-1) \mid g$;

(iv) $g - 1$ *is divisible by* $b^2 a^2 (a-1)$ *for some odd integers* $a, b > 1$, *with* $a \equiv 3 \pmod 4$.

Rank Four Abelian Groups

Proposition

The spectrum of abelian groups of rank 4 *is a subset of the integers satisfying conditions (i) or (ii) of the Main Theorem. Moreover, the spectrum of abelian groups of ranks* 3 *or* 4 *contains all the integers satisfying condition (ii) of the Main Theorem.*

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Proof: Notice that for an abelian group to have rank 4, it must have a subgroup isomorphic to Z_p^4 for some prime p.

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When $a \geq 5$, then $\sigma^0(A) = 1 + |A|$ for the rank 4 abelian group A. For $a = 3$, then $\sigma^0(A) = 1 + |A|$ or $\sigma^0(A) \equiv 1 \pmod 4$ for all except a few cases.

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For the exceptional cases with $a = 3$, $\sigma^0(A) \equiv 55 \pmod{81}$.

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For the exceptional cases with $a = 3$, $\sigma^0(A) \equiv 55 \pmod{81}$.

Conversely, all numbers g of the form $1 + p^4 n$ are the genus of groups of rank 3 or 4.

High Rank Abelian Groups

Let A be an abelian group of rank $n \geq 5$. So A has a subgroup isomorphic to Z_a^n. If a is even, then $\sigma^0(A) \equiv 1 \pmod 4$ and $\sigma^0(A) = \sigma^0(Z_2 \times Z_2 \times Z_{2n})$ for some n.

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If a is odd, then there is a rank four group B satisfying $|A| = |B|$ and so $\sigma^0(A) = \sigma^0(B)$.

Therefore, the genus spectrum is given by looking at the strong symmetric genus of groups of rank 3 or rank 4.

Number Theory

Let $\mathcal{S}_j$ denote those $g \in \mathcal{S}_A$ that satisfy the jth condition of the theorem but none of the earlier conditions. Let δ_j be the density of this set.

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$$\delta_1 = \frac{1}{4} + \frac{1}{81} - \frac{1}{324} = \frac{7}{27}.$$

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It is known that if $\gcd(a, q) = 1$, then

$$\sum_{\substack{n \le X \\ n \equiv a \pmod{q}}} \alpha_4(n) = C_q X + O(X^{1/3}).$$

Number Theory

We use this to show that the number $T(X)$ of integers $g \in \mathcal{S}_2$ with $g \equiv 2 \pmod 4$ and $g \le X$ is

$$T(X) = \frac{X}{4} - \sum_{\substack{h \le X \\ h \equiv 1 \pmod 4}} \alpha_4(h) = \frac{X}{4}\left(1 - \frac{16}{15\zeta(4)}\right) + O(X^{1/3}).$$

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Using a similar argument where $g \equiv 55 \pmod{81}$, we calculate the intersection with $\mathcal{S}_1$, and subtract it.

This gives the density of $\mathcal{S}_2$ is

$$\delta_2 = \frac{20}{27} - \frac{79}{100\zeta(4)} \approx 0.0108.$$

Number Theory

$\mathcal{S}_3$ can be described as the set $\mathcal{A}$ of integers g such that

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for some odd $a > 1$. The set $\mathcal{S}_4$ is similar.

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We prove that $\delta_3 \approx 0.0564$. and $\delta_4 \approx 0.0019$.

Altogether, we have

$$\delta(\mathcal{S}_A) = \delta_1 + \cdots + \delta_4 \approx 0.3284.$$

The Strong Symmetric Genus of Nilpotent Groups

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Clearly, all integers congruent to $1 \pmod 4$ are contained in $\mathcal{S}_N$.

$\sigma^o(Z_n \times D_4) = 2(n-1)$ for an odd integer n and so all integers congruent to $0 \pmod 4$ are contained in $\mathcal{S}_N$.

Other congruence classes

$\sigma^o(Z_n \times QD_4) = 2(2n-1)$ for an odd integer n and so all integers congruent to $2 \pmod 8$ are contained in $\mathcal{S}_N$.

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Theorem

Let $g \geq 0$. If g is not congruent to $6 \pmod 8$, then $g \in \mathcal{S}_N$.

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Let $g \geq 0$ with $g \equiv 6 \pmod 8$. If $g-1$ is prime, then $g \notin \mathcal{S}_N$.

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Let p be an odd prime, and let G be a nilpotent group of genus $\sigma^o(G) = 1 + p$. Then G is isomorphic to a direct product $O \times S_2$, where O is an abelian group of odd order that is either cyclic or $Z_p \times Z_{p^k}$ for some k and S_2 is a non-abelian 2-group with a cyclic subgroup of index 2.

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Since there are four families of groups, that can be S_2 and two possibilities for O, there are eight cases to consider.

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Therefore, $\underline{\delta}(\mathcal{S}_N) \geq \frac{8}{9}$.

Some Conclusions

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Corollary

There does not exist a congruence class consisting entirely of gaps in $\mathcal{S}_N$.

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May, C.L. and Zimmerman, J., The symmetric genus of finite abelian groups, Illinois J. of Math., Vol. 37, No. 3, Fall 1993, 400-423.

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Most of the differences hide in the $1 \pmod 4$ case.

Main Results

We were able to show that for an abelian group A, either $\sigma(A) \equiv 1 \pmod 4$ or $\sigma(A) = \sigma^o(A)$, unless the Sylow 2-subgroup is isomorphic to $Z_2 \times Z_{2^k}$ for some $k \geq 1$.

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If A has a Sylow 2-subgroup of rank 3 or higher then its genus is congruent to $1 \pmod 4$ and we can cover that case.

If A has cyclic Sylow 2-subgroup, then $\sigma(A) = \sigma^o(A)$.

Argument

Let $A \cong Z_{\beta_1} \times \cdots Z_{2\beta_{n-1}} \times Z_{2^k\beta_n}$, where all β_i are odd. Next, define $A_1 \cong Z_{\beta_1} \times \cdots Z_{\beta_{n-1}} \times Z_{2^{k+1}\beta_n}$.

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So $\sigma(A) = min\{\sigma^o(A), \sigma^o(A_1)\}$. Therefore, the symmetric genus spectrum is contained in the strong symmetric genus spectrum.

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