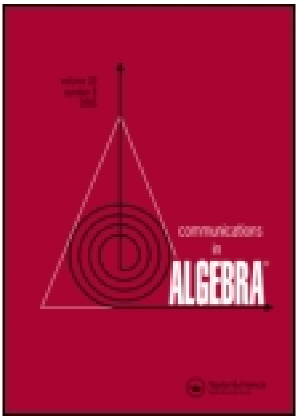

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The Symmetric Genus of p-Groups

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THE SYMMETRIC GENUS OF p-GROUPS

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Let G be a finite group. The symmetric genus $\sigma(G)$ is the minimum genus of any Riemann surface on which G acts. We show that a non-cyclic p-group G has symmetric genus not congruent to $1(mod\, p^3)$ if and only if G is in one of 10 families of groups. The genus formula for each of these 10 families of groups is determined. A consequence of this classification is that almost all positive integers that are the genus of a p-group are congruent to $1(mod\, p^3)$. Finally, the integers that occur as the symmetric genus of a p-group with Frattini-class 2 have density zero in the positive integers.

Key Words: Density; Fuchsian group; Frattini-class 2; *p*-Group; Symmetric genus; Riemann surface.

2010 Mathematics Subject Classification: Primary: 20H10; Secondary: 20D15; 20F38; 30F99; 57M60.

1. INTRODUCTION

Let G be a finite group. The *symmetric* genus $\sigma(G)$ of the group G is the minimum genus of any Riemann surface on which G acts faithfully (possibly reversing orientation). The symmetric genus is closely related to another parameter. The *strong symmetric* genus $\sigma^0(G)$ is the minimum genus of any Riemann surface on which G acts preserving orientation. The study of this classical parameter dates back to the work of Hurwitz and Burnside, among others; see [7] and [2, §289]. We use the modern terminology, which was introduced in [18].

Obviously, $\sigma(G) \leq \sigma^0(G)$ always, but in some (important) cases, the two parameters agree. If the group G does not have a subgroup of index 2, then G cannot act on a surface reversing orientation and thus $\sigma(G) = \sigma^0(G)$. In particular, if G is a group of odd order, then $\sigma(G) = \sigma^0(G)$.

A natural problem is to determine the positive integers that occur as the symmetric genus of a group (or a particular type of group). Indeed, whether or not there is a group of symmetric genus n for each value of the integer n remains a challenging open question; see the recent, important article [3]. Here we restrict our attention to p-groups, where p is an odd prime.

In considering the actions of a p-group on a surface, it is important to consider the exponent of the group. The following theorem [13, Th. 2] is important in this work.

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Theorem A. *Let p be an odd prime. Let G be a noncyclic group of order p^m that acts on a Riemann surface of genus $g \geq 2$. If the largest cyclic subgroup of G has index p^r, then the genus $g \equiv 1 \ (mod\, p^r)$.*

Theorem A may also be established by applying the main result of Kulkarni's important article [8, Th. 2.3]. Additional related work is in [9].

Fix the odd prime p, and let G be a group of order p^m that acts on a Riemann surface of genus $g \geq 2$. If G is not cyclic, then we always have $g \equiv 1 \ (mod\, p)$. Further, $g \equiv 1 \ (mod\, p^2)$ unless G contains an element of order p^{m-1}. There is a single family of non-abelian p-groups with this property for p an odd prime [4, §5.4]. We call these groups *quasiabelian* and let $QA(m)$ denote the group of order p^m. The genus formula for these groups was obtained in [13]. Also, the rank 2 abelian group $Z_p \times Z_{p^{m-1}}$ has symmetric genus 1 [5, p. 291]. We record the following, which was implicit in [13].

Theorem B. *Let p be an odd prime, and let G be a noncyclic p-group. Then the symmetric genus $\sigma(G) \equiv (1-p) \ (mod\, p^2)$ if and only if G is quasiabelian. If G is not quasiabelian, then $\sigma(G) \equiv 1 \ (mod\, p^2)$.*

In fact, $\sigma(G) \equiv 1 \ (mod\, p^3)$ unless G is in one of a few exceptional families, and these exceptional groups will be our focus here. A consequence of Theorem A is that if $\sigma(G) \not\equiv 1 \ (mod\, p^3)$, then G must contain an element of order p^{m-2} or larger. Then either G is a quasiabelian group or else G has exponent $Exp(G) = p^{m-2}$, that is, G has a cyclic subgroup of index p^2 but no cyclic subgroup of index p.

The families of p-groups with this property were classified, over a century ago, by Burnside [2] and Miller [14, 15]. A modern treatment of the classification was done by Ninomiya [16], and we use the notation of [16]. There are two abelian groups and 10 non-abelian groups of this type of order p^m, as long as $m \geq 6$. We classify the p-groups with symmetric genus not congruent to 1 $(mod\, p^3)$ to produce our first main result.

Theorem 1. *Let p be an odd prime, and let G be a noncyclic group of order p^m, with $m \geq 6$. Then $\sigma(G) \not\equiv 1 \ (mod\, p^3)$ if and only if G is isomorphic to one of the following ten groups: $Z_p \times Z_p \times Z_p^{m-2}$, $QA(m)$, $G_1(m)$, $G_3(m)$, $G_4(m)$, $G_5(m)$, $G_6(m)$, $G_7(m)$, $G_8(m)$, $G_{10}(m)$. If G is quasiabelian, then $\sigma(G) \equiv (1-p) \ (mod\, p^3)$. If G is isomorphic to a group in one of the remaining nine families, then $\sigma(G) \equiv (1-p^2) \ (mod\, p^3)$.*

We also consider the general problem of determining whether there is a p-group of symmetric genus g, for each value of $g \geq 2$, and interpret our results using the standard notion of density. Fix the odd prime p, and let J_p be the set of integers g for which there is a p-group of symmetric genus $g \geq 2$. Also, let K_p be the subset of J_p consisting of the integers that are not congruent to 1 $(mod\, p^3)$. The following two theorems are easy consequences of Theorem 1.

Theorem 2. *For each odd prime p, the set K_p has density 0 in the set of positive integers.*

Theorem 3. *Let p be an odd prime. Then almost all positive integers that are the symmetric genus of a p-group are congruent to* 1 $(mod\, p^3)$. *Further, the density* $\delta(J_p) \leq 1/p^3$.

We also consider the p-groups with Frattini-class 2. A p-group has Frattini-class 2, or Φ-class 2, in case its Frattini subgroup is central and elementary abelian. These groups are particularly important in group enumeration [1]. In his fundamental article [6], Higman obtains the lower bound for the number of p-groups of order p^k by counting the number of these groups that have Φ-class 2. However, the closeness of the upper bound for the number of p-groups of order p^k [1, Ch. 5] to this lower bound suggests that "most" p-groups have Φ-class 2. Here see Question 3 and its discussion in [12, p. 358]. In any case, it is clear that p-groups with Φ-class 2 are important in density considerations.

Fix the odd prime p, and let L_p be the set of integers $g \geq 2$ for which there is a p-group with Φ-class 2 of symmetric genus g. Then we prove that the set L_p has density 0 in the set of positive integers. This gives our final main result.

Theorem 4. *For each odd prime p, almost all positive integers that are the symmetric genus of a p-group with* Φ*-class* 2 *are in a set that has density zero in the positive integers.*

2. PRELIMINARIES

The groups of symmetric genus zero are the classical, well-known groups that act on the Riemann sphere (possibly reversing orientation) [5, §6.3.2]. The only odd order groups of symmetric group 0 are the cyclic groups.

If the odd order group G has symmetric genus 1, then G acts on a torus and preserves orientation. Consequently, G must have one of five partial presentations, or, in other words, G must be in one of the five Proulx classes (a) – (e) [5, p. 291]. Proulx classes (b), (d), and (e) contain generators of order 2 and so G is not an image of any of them. Proulx class (a) is a free abelian group of rank 2. Finally, if G is in Proulx classes (c), then $|G|$ is divisible by 3.

Let the finite group G act on the (compact) Riemann surface X of genus $g \geq 2$. Then represent $X = U/K$, where K is a surface group, and obtain a Fuchsian group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that $K = kernel\, \phi$. Associated with the Fuchsian group Γ are its signature and canonical presentation. Further, the non-euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature [17, p. 235]. Then the genus of the surface X on which G acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/4\pi. \tag{1}$$

The surface group K has no elements of finite order; see [10, p. 1198]. Consequently, the relations of Γ are "fulfilled" in G, that is, $A^\lambda = 1$ means that the element A has order λ and not merely a divisor of λ. Also, each period of Γ must divide $|G|$.

Next we quickly survey the Fuchsian groups with relatively small non-euclidean area and introduce some notation. First, an (ℓ, m, n) triangle group is a Fuchsian group Λ with signature

$$(0; +; [\ell, m, n]; \{\}), \text{ where } 1/\ell + 1/m + 1/n < 1.$$

If the group G is a quotient of Λ by a surface group, then G has a presentation of the form

$$X^{\ell} = Y^{m} = (XY)^{n} = 1. \tag{2}$$

We will say that G has partial presentation $T(\ell,\ m,\ n)$ (with the relations fulfilled) or just a partial presentation of type T.

An $(\ell,\ m,\ n,\ t)$ quadrilateral group is a Fuchsian group Λ with signature

$$(0; +; [\ell, m, n, t]; \{\ \}), \text{ where } 1/\ell + 1/m + 1/n + 1/t < 2.$$

A quotient group G of Λ has a presentation of the form

$$X^{\ell} = Y^{m} = Z^{n} = (XYZ)^{t} = 1 \tag{3}$$

We will denote this partial presentation $Q(\ell,\ m,\ n,\ t)$.

Finally, let Λ be an NEC group with signature

$$(1; +; [k]; \{\ \})$$

Then the quotient space U/Λ is a torus; the non-euclidean area is $\mu(\Lambda)/2\pi = 1 - 1/k$. A quotient group G of Λ has two generators C and D with the single relation

$$[C, D]^{k} = 1, \tag{4}$$

for some $k \geq 2$, and we will say G has partial presentation $A(k)$.

For each odd prime p, the family of quasiabelian groups $QA(m)$ of order p^{m} will be important here. For $m \geq 3$, $QA(m)$ is the group with generators a, b and defining relations

$$a^{p^{m-1}} = b^{p} = 1, b^{-1}ab = a^{1+p^{m-2}}. \tag{5}$$

The symmetric genus is given by $\sigma(QA(m)) = (p-1)(p^{m-1}-2)/2$, and $QA(m)$ has partial presentation $T(p,\ p^{m-1},\ p^{m-1})$ [13, Th. 2].

3. ACTIONS OF p-GROUPS

Now we obtain some general results about the possible types of partial presentations for a p-group acting on a surface, where p is an odd prime. To obtain our first result of this type, it is only necessary to carefully consider the possible signatures for a Fuchsian group Γ with $\mu(\Gamma)/2\pi < 2 - 2/p$ such that there is a surface group K that is a normal subgroup of Γ (so that the quotient group Γ/K acts on a Riemann surface X). If g is the genus of X, then from (1) we have $g < 1 + |G| \cdot (1 - 1/p)$ or, alternately, $|G| > (g-1)$. The check of the signatures yields the following necessary condition.

Proposition 1. *Let p be an odd prime, and let G be a p-group with $\sigma(G) \geq 2$. If $\sigma(G) < 1 + |G| \cdot (1 - 1/p)$, then G has a partial presentation of type T, Q or $A(k)$. Further, if the partial presentation is of type Q, then at least two periods are equal to p.*

The converse of Proposition 1 also holds if appropriate restrictions are placed on the parameters in the presentation so that the associated Fuchsian group Γ has $\mu(\Gamma)/2\pi < 2 - 2/p$ and, further, the kernel of the homomorphism from Γ onto G is required to be a surface group.

More refined versions of Proposition 1 can be obtained, of course, by reducing the allowed area and thus making the group action "larger." We will need two refinements, and we give these here. The proof only requires a careful checking of possible signatures.

Proposition 2. *Let G be a p-group with $\sigma(G) \geq 2$. Then the following statements hold*:

(1) $\sigma(G) < 1 + (1 - 1/p^2)|G|/2$ *if and only if G has a partial presentation $T(p, m, n)$ with $m \leq n$ (and $n > 3$ if $p = 3$), $T(p^2, m, n)$ with $p^2 \leq m \leq n$, $A(p)$ or $Q(3, 3, 3, 3)$ (if $p = 3$)*;
(2) $\sigma(G) < 1 + (1 - 1/p)|G|/2$ *if and only if G has a partial presentation $T(p, m, n)$ with $m \leq n$ (and $n > 3$ if $p = 3$).*

Certain subgroups of the p-group G are especially useful when considering the types of partial presentations G can have. For $i \geq 1$, let Ω_i be the characteristic subgroup of G generated by elements of order p^i or less. In general, if Ω_i is a proper subgroup of G for some i, then G must have at least one generator in each generating set that has order p^{i+1} or higher. For example, if the prime $p > 3$ and $\Omega_1 \neq G$, then G cannot have a partial presentation of type $T(p, p, t)$. In general, suppose $\Omega_i \neq G$ and G has a partial presentation of type T or Q so that G is a quotient of Γ, where Γ is a triangle group or a quadrilateral group, respectively. Then, just from the presentations (2) and (3), at least two of the periods in the signature of Γ must be at least p^{i+1}.

As we shall see, of particular relevance here is the case $i = m - 3$, and we have the following improvement of Proposition 2 for p-groups of this type with $Exp(G) = p^{m-2}$.

Proposition 3. *Let G be a group of order p^m with $\sigma(G) \geq 2$. Suppose $Exp(G) = p^{m-2}$ and $\Omega_{m-3} \neq G$. Then the following statements hold*:

(1) $\sigma(G) < 1 + (1 - 1/p^2)|G|/2$ *if and only if G has a partial presentation $T(p, p^{m-2}, p^{m-2})$, $T(p^2, p^{m-2}, p^{m-2})$, or $A(p)$*;
(2) $\sigma(G) < 1 + (1 - 1/p)|G|/2$ *if and only if G has a partial presentation $T(p, p^{m-2}, p^{m-2})$. In this case, $\sigma(G) = 1 - p^2 + (p - 1)p^{m-1}/2$.*

It is also possible to obtain results of this type for p-groups that have rank 3. In this case the group cannot have presentations of either type T or A. If, further, Ω_{m-3} is a proper subgroup, then we have the following improvement of Proposition 1, again for groups with $Exp(G) = p^{m-2}$.

Proposition 4. *Let G be a group of order p^m with genus $\sigma(G) \geq 2$. Suppose G has rank 3, and, further, $Exp(G) = p^{m-2}$ and $\Omega_{m-3} \neq G$. Then $\sigma(G) < 1 + |G| \cdot (1 - 1/p)$ if and only if G has partial presentation type $Q(p, p, p^{m-2}, p^{m-2})$. In this case, $\sigma(G) = 1 - p^2 + (p - 1)p^{m-1}$.*

Now we consider the consequences of Theorem A. Let the noncyclic group G have order p^m, where p is an odd prime. The only p-group such that the genus $\sigma(G)$ is not congruent to 1 $(mod\, p^2)$ is the quasiabelian group. If $G = QA(m)$, then $\sigma(G) = (p-1)(p^{m-1}-2)/2$ [13, Th. 2]; in this case, $\sigma(G) \equiv (1-p)$ $(mod\, p^2)$. This establishes Theorem B.

In fact, $g \equiv 1$ $(mod\, p^3)$ unless G is one of a few exceptional families, and these groups will be our focus here. The following theorem is basic.

Theorem 5. *Let G be a non-cyclic group of order p^m. If $\sigma(G)$ is not congruent to 1 $(mod\, p^3)$, then either $G \cong QA(m)$ or $Exp(G) = p^{m-2}$.*

Proof. By Theorem A, $Exp(G)$ must be at least p^{m-2}. If G contains an element of order p^{m-1}, then, again, the only possibility is that $G \cong QA(m)$. In this case, $\sigma(G) = (p-1)(p^{m-1}-2)/2 \equiv (1-p)$ $(mod\, p^3)$. The only remaining possibility is that $Exp(G) = p^{m-2}$.

4. *P*-GROUPS WITH CYCLIC SUBGROUPS OF INDEX p^2

Here we consider the families of p-groups that have cyclic subgroups of index p^2 but no cyclic subgroup of index p. The classification of these groups was accomplished long ago by Burnside [2] and Miller [14, 15]. This classification is also in [16].

It is well-known that as m increases, the number of groups of order p^m increases in a dramatic way [12]. Further, this number increases as the odd prime p increases [12]. It is interesting, though, that there are a small, fixed number of these p-groups with exponent equal to p^2, independent of both m and p. There are two abelian groups and 10 non-abelian groups of this type, as long as $m \geq 6$. For the ten non-abelian groups, we use the notation of [16]. For convenience, we give the presentations of these groups in Table 1. For non-abelian groups of order p^m with $3 \leq m \leq 5$, there are fewer groups, although with one exception, they are all in the families indicated in Table 1. The exception is a group of order 81, G_{11}, that is not in any of the families [16]; this group has genus $28 \equiv 1$ $(mod\, 27)$. There is one group of order p^3 (G_1), seven groups of order p^4 for $p \neq 3$, and eight groups of order 81 (all but the groups G_8, G_9 and G_{10}) and nine groups of order p^5 (all but G_{10}).

Table 1 Non-abelian groups with cyclic subgroup of index p^2

$G_1 = \langle a, b, c \mid a^{p^{m-2}} = b^p = c^p = 1, ab = ba, c^{-1}ac = ab, bc = cb \rangle.$
$G_2 = \langle a, b \mid a^{p^{m-2}} = b^{p^2} = 1, b^{-1}ab = a^{1+p^{m-3}} \rangle.$
$G_3 = \langle a, b, c \mid a^{p^{m-2}} = b^p = c^p = 1, b^{-1}ab = a^{1+p^{m-3}}, ac = ca, bc = cb \rangle$
$G_4 = \langle a, b, c \mid a^{p^{m-2}} = b^p = c^p = 1, ab = ba, ac = ca, c^{-1}bc = a^{m-3+p^{m-3}} b \rangle.$
$G_5 = \langle a, b, c \mid a^{p^{m-2}} = b^p = c^p = 1, ab = ba, c^{-1}ac = ab, c^{-1}bc = a^{m-3+p^{m-3}} b \rangle.$
$G_6 = \langle a, b, c \mid a^{p^{m-2}} = b^p = c^p = 1, ab = ba, c^{-1}ac = ab, c^{-1}bc = a^{m-3+rp^{m-3}} b \rangle.$
$G_7 = \langle a, b, c \mid a^{p^{m-2}} = b^p = c^p = 1, b^{-1}ab = a^{1+p^{m-3}}, c^{-1}ac = ab, bc = cb \rangle.$
$G_8 = \langle a, b \mid a^{p^{m-2}} = b^{p^2} = 1, b^{-1}ab = a^{1+p^{m-4}} \rangle.$
$G_9 = \langle a, b \mid a^{p^{m-2}} = b^{p^2} = 1, a^{-1}ba = b^{1+p} \rangle.$
$G_{10} = \langle a, b \mid a^{p^{m-2}} = 1, a^{p^{m-3}} = b^{p^2}, a^{-1}ba = b^{1-p} \rangle.$

Finally, if G is an abelian group of this type, then G is isomorphic to $Z_{p^2} \times Z_{p^{m-2}}$ or $Z_p \times Z_p \times Z_{p^{m-2}}$. Each rank 2 abelian group $Z_{p^2} \times Z_{p^{m-2}}$ has symmetric genus 1 [5, p. 291], and $\sigma(Z_p \times Z_p \times Z_{p^{m-2}}) = (p-1)p^{m-1} - p^2 + 1 \equiv (1-p^2)\ (mod\, p^3)$ [11, Th. 4].

The following theorem will be quite helpful.

Theorem 6. *Let p be an odd prime, and let $G_n(m)$ be a non-abelian p-group of order p^m and $Exp(G) = p^{m-2}$. (The range of m varies with the family.) Then $\Omega_{m-3}(G) \neq G$ for all p and all m, with two exceptions; the exceptions are the groups $G_6(4)$ and $G_7(4)$ with $p = 3$.*

Proof. The proof of this result for each of the 10 infinite families of groups is the same. First, define the subgroup H in each group $G_n(m)$ generated by all powers of the generators from Table 1 that have order p^{m-3} or less. This results in a subgroup of index p in all cases except $G_1(3)$ and $G_2(4)$. The theorem is true in these two exceptional cases. Clearly, $H \subseteq \Omega_{m-3}(G)$. Finally, an induction argument is used to find a formula for a power of each element not in H. Using this formula, we show that each element not in H has order p^{m-2}. This proves that $H = \Omega_{m-3}(G)$ and so $\Omega_{m-3}(G) \neq G$. We will sketch the arguments for $G_5(m)$, $G_6(m)$, and $G_7(m)$. The arguments for all other families prove the result without exceptional cases.

The infinite families $G_5(m)$ and $G_6(m)$ have the same presentations, except for the relation $c^{-1}bc = a^{rp^{m-3}}b$ with different values of r. In $G_5(m)$, the value of r is 1 and for $G_6(m)$, the value of r is a quadratic nonresidue modulo p. Using induction, we find a formula for $(a^j b^k c^l)^n$. Substituting p for n, we can eliminate the powers of b and c. The equation that remains is

$$(a^j b^k c^l)^p = a^{pj - \binom{p}{2}klrp^{m-3} + \binom{p+1}{3}jl^2rp^{m-3} - \binom{p}{2}\binom{l}{2}jrp^{m-3}}. \tag{6}$$

Now $\binom{p}{2}$ is always a multiple of p and $\binom{p+1}{3}$ is a multiple of p for $p \geq 5$. Therefore, for $p \geq 5$, the order of $a^j b^k c^l$ equals the order of a^j, which is p^{m-2} as long as p does not divide j. Now for $p = 3$, the equation reduces to

$$(a^j b^k c^l)^3 = a^{3j + 4jl^2r3^{m-3}}. \tag{7}$$

If $m > 4$, the order of $a^j b^k c^l$ is 3^{m-2}. Finally, if $m = 4$, namely G has order 81, the order of $a^j b^k c^l$ is 9 if and only if 3 does not divide $j(1 + 4rl^2)$. Since $a^j b^k c^l \notin H$, we have that 3 does not divide j. In $G_5(4)$, $r = 1$, and so the order of $a^j b^k c^l$ is 9. In $G_6(4)$, $r = 2$, and so there is an element of order 3 that is not in H. Therefore, $\Omega_{m-3}(G_5(m)) \neq G_5(m)$ and $\Omega_{m-3}(G_6(m)) \neq G_6(m)$, except for $p = 3$ and $m = 4$. These calculations show that the 3-group $G_6(4)$ has genus one.

The infinite family, $G_7(m)$ has a complicated formula for the power of any element. The following equation gives the p^{th} power of an arbitrary element, using the facts that $\binom{p}{2} \equiv 0\ (mod\, p)$ and $b^p = c^p = 1$:

$$(a^j b^k c^l)^p = a^{pj + s_j(p-1)lp^{m-3}} \tag{8}$$

The function $s_j(n)$ is defined by $s_j(n) = \sum_{i=1}^{n} \binom{ij}{2}$. For elements not in H, we have that p does not divide j and so $j \neq 0$. If $p \neq 3$, we have $s_j(p-1) \equiv 0\ (mod\, p)$ and therefore $o(a^jb^kc^l) = p^{m-2}$. If $p = 3$ and $m \geq 5$, then $(a^jb^kc^l)^3 = a^{3j+t3^{m-3}}$, for some integer t, and so $o(a^jb^kc^l) = 3^{m-2}$. Therefore, $\Omega_{m-3}(G_7(m)) \neq G_7(m)$, except when $p = 3$ and $m = 4$. For $p = 3$, the group of order 81, $G_7(4)$ is generated by two elements of order 3.

To establish Theorem 1, we determine the genus of each group in each of the ten infinite families; the genus of each quasiabelian group and each rank 3 abelian group is already known. Clearly, none of these groups have symmetric genus zero. Also, none of these groups act on the torus, with a single exception.

Theorem 7. *Let p be an odd prime, and let G be a non-abelian group of order p^m, with $m \geq 4$ and $Exp(G) = p^{m-2}$. Then $\sigma(G) = 1$ if and only if $p = 3$ and $G \cong G_6(4)$.*

Proof. As stated in the proof of Theorem 6, the 3-group $G_6(4)$ is in class (c) and thus has symmetric genus one. The elements c and ca of order 3 generate this group and their product c^2a also has order 3.

Now let G be a non-abelian group of order p^m with $Exp(G) = p^{m-2}$ and assume $\sigma(G) = 1$. If p is an odd prime, then G must be in Proulx class (c). However, if a non-abelian p-group is in class (c), then the prime is $p = 3$. A 3-group in class (c) is generated by elements of order 3 and then $\Omega_1(G) = G$. But since $m \geq 4$, we have $\Omega_1 \subseteq \Omega_{m-3}$ Thus, $\Omega_1 \neq G$ for all G, with the two exceptions $G_6(4)$ and $G_7(4)$. A computer search of the elements of the 3-group $G_7(4)$ reveals that while it is generated by many sets of two elements of order 3, the product of these elements has order 9.

In the proof of Theorem 1, it is natural to consider these groups in several sets of groups. Each set of groups contains groups with similar presentations and the same genus formula. First, if G is the quasiabelian group of order p^m, we already know that $\sigma(G) = (p-1)(p^{m-1}-2)/2 \equiv (1-p)\ (mod\, p^3)$. In the remainder of the proof, the prime p will be a fixed odd prime.

Let $G = G_4(m)$. Then $G' = \langle a^{p^{m-3}} \rangle$ and the commutator quotient group $G/G' \cong Z_p \times Z_p \times Z_{p^{m-3}}$. Thus, G has rank 3. Then G is generated by b, c, and a, with $o(bca) = p^{m-2}$. Hence G has partial presentation $Q(p, p, p^{m-2}, p^{m-2})$. Now by Theorem 6 and Proposition 4, $\sigma(G) = (p-1)p^{m-1} - p^2 + 1 \equiv (1-p^2)\ (mod\, p^3)$. Essentially, the same argument can be used for $G_3(m) \cong Z_p \times QA(m-1)$; this rank 3 group also has genus $(p-1)p^{m-1} - p^2 + 1$.

Next let $G = G_1(m), G_5(m), G_6(m)$, or $G_7(m)$. For each of these groups, $G = \langle c, a \rangle$. Since $c \in \Omega_{m-3}(G)$ and $a \notin \Omega_{m-3}(G)$, we have that $ca \notin \Omega_{m-3}(G)$ and by Theorem 6, the order of ca is p^{m-2}. Therefore, all of these families of groups have presentation $T(p, p^{m-2}, p^{m-2})$ and by Proposition 3 (b), $\sigma(G) = 1 - p^2 + (p-1)p^{m-1}/2$.

Now we consider the groups $G = G_2(m)$ or $G_9(m)$. In both cases, $\Omega_1(G) \subseteq \Phi(G)$, and so all elements of order p are non-generators. Hence G does not have presentation $T(p, p^{m-2}, p^{m-2})$. It is clear from the presentation in Table 1 that G has presentation $A(p)$, and this presentation gives the genus action by Proposition 3 (a). Thus $\sigma(G) = (p-1)p^{m-1}/2 + 1 \equiv 1\ (mod\, p^3)$.

Finally, let $G = G_8(m)$ or $G_{10}(m)$. Again, $\Omega_1(G) \subseteq \Phi(G)$ so that G does not have presentation $T(p, p^{m-2}, p^{m-2})$. If G had presentation $A(p)$, then the quotient

Table 2 Symmetric genus of the relevant families

Family	Genus Formula
QA	$(p-1)(p^{m-1}-2)/2$
$Z_p \times Z_p \times Z_{p^{m-2}}, G_3, G_4$	$(p-1)p^{m-1}-p^2+1$
G_1, G_5, G_6, G_7	$((p-1)p^{m-1})/2-p^2+1$
G_8, G_{10}	$((p^2-1)p^{m-2})/2-p^2+1$
G_2, G_9	$((p-1)p^{m-1})/2+1$

G/Ω_1 would be abelian, but it is easy to see that this is not the case. Further, G has presentation $T(p^2, p^{m-2}, p^{m-2})$. This is clear for G_8; an element of order p^2 in G_{10} is $a^{p^{m-5}}b^{-1}$. Now by Proposition 3 (a), $\sigma(G) = (p^2-1)p^{m-2}/2 - p^2 + 1$.

This finishes the genus calculation of the p-groups with genus not congruent to 1 $(mod\, p^3)$; this classification is given by Theorem 1. Table 2 gives the formula for the symmetric genus for each of these groups (and also the exceptions G_2 and G_9).

The genus formulas in Table 2 are correct for groups of orders p^m for $3 \le m \le 5$ and all odd primes p, with three exceptions for $p = 3$. The 3-groups from the families $G_1(3)$, $G_6(4)$, and $G_7(4)$ have genus 1, 1, and 10, respectively. Further, the congruence statements asserted in Theorem 1 also hold for $m = 4$ and $m = 5$ and all odd primes p. Furthermore, the congruence statements in Theorem 1 do not hold for the $m = 3$ case.

5. DENSITY

Now we turn our attention to the general problem of determining whether there is a p-group of symmetric genus g, for each value of the integer g. One way to describe the results involves the standard notion of density. Kulkarni described one of his results in this way [8, p. 201].

Let p be an odd prime, and let J_p be the set of integers $g \ge 2$ for which there is a p-group of symmetric genus g. For an integer n, let $f_p(n)$ denote the number of integers in J_p that are less than or equal to n. Then the natural density $\delta(J_p)$ of J_p in the set of positive integers is

$$\delta(J_p) = \lim_{n\to\infty} \frac{f(n)}{n}.$$

Also, let K_p be the subset of J_p consisting of the integers that are not congruent to 1 modulo p^3, with the companion "counting" function denoted h_p. We use Theorem 1 to bound the values of the function h_p.

Proof of Theorem 2. Assume $p > 3$, so that $h_p(p^3) = 3$. Write $n = p^m$, with $m \ge 4$. Theorem 1 lists the groups with genus in the set K_p, with the values detailed in Table 2. Let G be one of these p-groups. Then from the basic lower bound for the genus of a p-group, we have

$$|G| \le 2p(\sigma(G) - 1)/(p-3) \le 2p(n-1)/(p-3). \tag{9}$$

But $2p/(p-3) \le p$ so that $|G| \le p(n-1) \le p(p^m - 1)$. Thus $|G| \le p^m$. Now, in each family, the only possible groups with genus in the range are those of order p^4 (if the group is defined), p^5 (if the group is defined), $p^6, \ldots, p^m$, that is, in each family, there are at most $m-3$ groups with genus in the range. However, for these groups of a particular order, there are exactly 4 distinct values, for the first 4 sets of groups in Table 2. Further, there is at least one group in each set defined at order p^5 and in each set except the last set containing G_8, G_{10} at order p^4. Also, it is elementary to see that the genus values for different orders do not overlap. Hence, the value of the counting function is $h_p(n) = h_p(p^m) = 4(m-4) + 3 + 3$. Thus $\delta(K_p) = 0$.

The proof for $p = 3$ is very similar, using the basic lower bound $|G| \leq 9[\sigma(G) - 1]$ for the genus of a 3-group, and we omit the details.

It is important to note that all that is needed to establish Theorem 2 is the classification of the p-groups that have cyclic subgroups of index p^2 but no cyclic subgroup of index p. The key is really the fact that there are a fixed, finite number of groups of order p^m that have exponent p^{m-2}, independent of the odd prime p and the exponent m.

Theorem 2 clearly implies Theorem 3. The problem of determining the density of the set J_p still remains, of course, and only the integers congruent to 1 $(mod\, p^3)$ need be considered.

6. FRATTINI-CLASS 2

Now we turn our attention to the p-groups with Frattini-class 2. A p-group G has Frattini-class 2, or Φ-class 2, in case its Frattini subgroup Φ is central and elementary abelian (but not trivial).

Let p be an odd prime, and let G be a group of order p^m with Φ-class 2. Then the exponent of G is either p or p^2 (and both are possible). But the group G has no elements of order p^3, because each square is in Φ and G/Φ is elementary abelian. Hence $Exp(G) \leq p^2$.

The basic upper bound [19] for the order of a p-group acting on a Riemann surface produces a lower bound for the symmetric genus of a p-group. In this respect, the prime 3 is special. Let the p-group G act on a Riemann surface of genus $g \geq 2$. If $p = 3$, then $|G| \leq 9(g-1)$; if $p \geq 5$, then $|G| \leq 2p(g-1)/(p-3)$ [19, Th. 1.1.2]. These bounds were also established in [8, Cor. 3.5]; they yield the following lemma.

Lemma 1. *Let G be a group of order 3^m, with symmetric genus $\sigma(G) \geq 2$. Then*

$$\sigma(G) \geq 3^{m-2} + 1.$$

Lemma 2. *Let p be a prime, $p \geq 5$, and let G be a group of order p^m, with symmetric genus $\sigma(G) \geq 2$. Then*

$$\sigma(G) \geq (p-3)p^{m-1}/2 + 1.$$

Next we find an upper bound for the genus of a p-group with Φ-class 2.

Lemma 3. *Let p be an odd prime, and let G be a group of order p^m, with $m \geq 3$. If G has Φ-class 2, then*

$$\sigma(G) \leq 1 + p^{m-2}[(p^2-1)m - 2p^2]/2. \tag{10}$$

Proof. The rank r of the p-group G is at most $m-1$. Thus G has a presentation with r generators, each of which has order p or p^2, and $r \leq m-1$. The group G is a quotient of a Fuchsian group with $r+1$ periods (r periods equal to the orders of the generators and the last period equal to the order of the product of the generators). Since each period is at most p^2, we obtain the following bound:

$$\sigma(G) \leq 1 + p^m[-2 + m(1 - 1/p^2)]/2.$$

Hence the bound (10) holds.

Applying Theorem A to p-groups with Φ-class 2 yields the following lemma.

Lemma 4. *Let p be an odd prime, and let G be a group of order p^m, with $m \geq 3$. If G has Φ-class 2, then*

$$\sigma(G) \equiv \ 1 \ mod \ p^{m-2}. \tag{11}$$

Proof. The group G is not cyclic, since $|G|$ is at least p^3 and G has Φ - class 2. If $\sigma(G) = 1$, then (11) obviously holds. Assume, then, that $\sigma(G) \geq 2$, and let G act on a Riemann surface of genus $\sigma(G) \geq 2$. Since $Exp(G)$ is either p or p^2, the largest cyclic subgroup of G has index p^{m-1} or p^{m-2}. In either case, applying Theorem A gives (11).

Fix the odd prime p, and let L_p be the set of integers $g \geq 2$ for which there is a p-group G with Φ-class 2 of such that the symmetric genus $\sigma(G) = g$. For an integer n, let $f_p(n)$ denote the number of integers in L_p that are less than or equal to n. We use Lemma 4 to obtain an upper bound for $f_p(n)$, with $n = 2^k$.

Theorem 8. *The set L_p has density 0 in the set of positive integers.*

Proof. First, if G is a p-group with Φ-class 2 and $|G| \leq p^2$, then G has symmetric genus zero or one and does not contribute to the set L_p.

Now let $n = p^k$, with $k \geq 3$, and assume $p > 3$. We need to count the p-groups with Φ-class 2 that have genus n or less. Let G be one of these p-groups. From the bound (9), just as in the proof of Theorem 2, we have $|G| \leq p^k$. Write $|G| = p^m$, with $3 \leq m \leq k$. Combining Lemmas 2, 3 and 4, we see that $\sigma(G)$ must be in an arithmetic sequence with the common difference $d = p^{m-2}$. Let ℓ be the number of terms in this sequence. The first term $a_1 = 1 + p(p-3)d/2$, and the last term $a_\ell = 1 + d[(p^2-1)m - 2p^2]/2$. Then a simple calculation shows that the number of terms $\ell = (p^2-1)m/2 - 3(p^2-p)/2 + 1$. Now write $b_p = (p^2-1)/2$ and $c_p = 1 - 3(p^2-p)/2$. Thus if $|G| = p^m$, then there are only $b_p \cdot m + c_p$ possibilities for $\sigma(G)$. Since $3 \leq m \leq k$, the total number of possibilities for $\sigma(G)$ (ignoring the overlapping in the sequences) is therefore bounded by the sum

$$\sum_{m=3}^{k} \left(b_p \cdot m + c_p\right) = b_p \left[\frac{k(k+1)}{2} - 3\right] + c_p(k-2). \tag{12}$$

Now letting $n = p^k \to \infty$ gives $\delta(L_p) = 0$.

The proof for $p = 3$ is very similar, and we omit the details. Combining Lemmas 1, 3, and 4, we see that $\sigma(G)$ must be in an arithmetic sequence with the common difference $d = 3^{m-2}$. We compute the expression bounding the value $h(3^k)$ and again show that it is is $o(k^2)$.

Theorem 3 is just a restatement of Theorem 8, of course.

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