THE GENUS SPECTRUM OF FINITE ABELIAN GROUPS

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Definitions

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The *real genus* $\rho(G)$ is the minimum algebraic genus of any compact bordered Klein surface on which G acts faithfully.

A Natural Question

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A natural problem is to determine the positive integers that occur as the strong symmetric genus of a group (or a particular type of group). This set is called the strong symmetric genus spectrum.

May & Zimmerman, 2003. There is a group of strong symmetric genus n for each value of the integer n.

Upper and Lower Density

Let A be a set of positive integers. For an integer X, let $[1, X]$ be the set of integers between 1 and X and define $A(X) = |A \cap [1, X]|$.

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If A is a set of integers, its *lower* and *upper asymptotic densities*, denoted $\underline{\delta}(A)$ and $\overline{\delta}(A)$, are given by

$$\underline{\delta}(A) = \liminf_{X \to \infty} \frac{A(X)}{X}$$

and

$$\overline{\delta}(A) = \limsup_{X \to \infty} \frac{A(X)}{X}.$$

Density in the Integers

A set A is said to have an *asymptotic density*, if $\underline{\delta}(A) = \overline{\delta}(A)$; when A does have an asymptotic density, it is denoted $\delta(A)$.

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Clearly the 2003 result of May and Zimmerman is considerably stronger than the above density statement.

Density of the Abelian Spectrum

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We will also give necessary and sufficient conditions for a positive integer g to be the strong symmetric genus of an abelian group.

Formulas I

Recall that every finite abelian group G has a canonical representation $G \cong Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}$, with standard invariants $m_1, m_2, \ldots, m_r$ subject to $m_1 > 1$ and $m_i | m_{i+1}$ for $1 \leq i < r$.

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Maclachlan (1965) proved that if G is an abelian group of rank $r \geq 3$, with $|G| \geq 10$, then

$$\sigma^0(G) = 1 + \frac{|G|}{2} \min_{0 \leq \gamma \leq r/2} \left\{ 2\gamma - 2 + \sum_{i=1}^{r-2\gamma} \left(1 - \frac{1}{m_i}\right) + \left(1 - \frac{1}{m_{r-2\gamma}}\right) \right\}. \tag{1}$$

Formulas II

For example, when $a > 1$ and $a^3 n \geq 10$, Maclachlan's formula yields

$$\sigma^0(\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}) = 1 - a^2 + a^2(a-1)n.$$

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In particular, when $a = 2$, this reveals that S contains the entire residue class $g \equiv 1 \pmod 4$.

Also when a is odd, $g \equiv 1 - a^2 \pmod{a^2(a-1)}$ and this is equivalent to $g - 1$ is divisible by a^2 for some odd integer a with $(a-1)|g$ by the Chinese Remainder Theorem.

Formulas III

When $b \geq 2$ and $bn > 2$, Maclachlan's formula gives

$$\sigma^0(Z_a \times Z_{ab} \times Z_{abn}) = 1 + b^2 a^2 (a-1) n. \tag{2}$$

In all cases, except when a and b are odd, with $a \equiv 3 \pmod 4$, $g \equiv 1 \pmod 4$.

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In all cases, except when a and b are odd, with $a \equiv 3 \pmod 4$, $g \equiv 1 \pmod 4$.

Proposition

The spectrum of abelian groups of rank 3 *consists of the congruence class* $g \equiv 1 \pmod 4$ *and the integers* g *satisfying conditions (iii) or (iv) of the Theorem below.*

Main Theorem

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Let $g \geq 2$*. Then* $g \in S$ *if and only if* g *satisfies one of the following conditions:*

(i) $g \equiv 1 \pmod 4$ *or* $g \equiv 55 \pmod{81}$*;*

(ii) $g - 1$ *is divisible by* p^4 *for some odd prime* p*;*

(iii) $g - 1$ *is divisible by* a^2 *for some odd integer* a *with* $(a-1) \mid g$*;*

(iv) $g - 1$ *is divisible by* $b^2 a^2 (a-1)$ *for some odd integers* $a, b > 1$*, with* $a \equiv 3 \pmod 4$*.*

Rank Four Abelian Groups

Proposition

The spectrum of abelian groups of rank 4 *is a subset of the integers satisfying conditions (i) or (ii) of the Main Theorem. Moreover, the spectrum of abelian groups of ranks* 3 *or* 4 *contains all the integers satisfying condition (ii) of the Main Theorem.*

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Proposition

The spectrum of abelian groups of rank 4 is a subset of the integers satisfying conditions (i) or (ii) of the Main Theorem. Moreover, the spectrum of abelian groups of ranks 3 or 4 contains all the integers satisfying condition (ii) of the Main Theorem.

Proof: Notice that for an abelian group to have rank 4, it must have a subgroup isomorphic to Z_p^4 for some prime p.

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When $a \geq 5$, then $\sigma^0(A) = 1 + |A|$ for the rank 4 abelian group A. For $a = 3$, then $\sigma^0(A) = 1 + |A|$ or $\sigma^0(A) \equiv 1 \pmod 4$ for all except a few cases.

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For the exceptional cases with $a = 3$, $\sigma^0(A) \equiv 55 \pmod{81}$.

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For the exceptional cases with $a = 3$, $\sigma^0(A) \equiv 55 \pmod{81}$.

Conversely, all numbers g of the form $1 + p^4 n$ are the genus of groups of rank 3 or 4.

High Rank Abelian Groups

Let A be an abelian group of rank $n \geq 5$. So A has a subgroup isomorphic to Z_a^n. If a is even, then $\sigma^0(A) \equiv 1 \pmod 4$ and $\sigma^0(A) = \sigma^0(Z_2 \times Z_2 \times Z_{2n})$ for some n.

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Therefore, the genus spectrum is given by looking at the strong symmetric genus of groups of rank 3 or rank 4.

Number Theory

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$$\delta_1 = \frac{1}{4} + \frac{1}{81} - \frac{1}{324} = \frac{7}{27}.$$

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It is known that if $\gcd(a, q) = 1$, then

$$\sum_{\substack{n \leq X \\ n \equiv a \pmod{q}}} \alpha_4(n) = C_q X + O(X^{1/3}).$$

Number Theory

We use this to show that the number $T(X)$ of integers $g \in \mathcal{S}_2$ with $g \equiv 2 \pmod 4$ and $g \le X$ is

$$T(X) = \frac{X}{4} - \sum_{\substack{h \le X \\ h \equiv 1 \pmod 4}} \alpha_4(h) = \frac{X}{4}\left(1 - \frac{16}{15\zeta(4)}\right) + O(X^{1/3}).$$

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Using a similar argument where $g \equiv 55 \pmod{81}$, we calculate the intersection with $\mathcal{S}_1$, and subtract it.

This gives the density of $\mathcal{S}_2$ is

$$\delta_2 = \frac{20}{27} - \frac{79}{100\zeta(4)} \approx 0.0108.$$

Number Theory

$\mathcal{S}_3$ can be described as the set $\mathcal{A}$ of integers g such that

$$g \equiv 1 - a^2 \pmod{a^2(a-1)} \qquad (*)$$

for some odd $a > 1$. The set $\mathcal{S}_4$ is similar.

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We prove that $\delta_3 \approx 0.0564$. and $\delta_4 \approx 0.0019$.

Altogether, we have

$$\delta(\mathcal{S}) = \delta_1 + \cdots + \delta_4 \approx 0.3284.$$

The Real Genus

Every finite abelian group G has a representation $G \cong Z_{e_1} \times \cdots Z_{e_m} \times Z_{d_1} \times \cdots \times Z_{d_\ell} \times Z_2^n$, where e_i is a multiple of 4 $d_j \geq 3$ is odd and as we move from left to right all invariants divide the previous one.

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McCullough (1990) obtained the formulas for the real genus of abelian non-cyclic groups by using graph theoretic techniques. One of 4 such formulas is given below.

$$\rho(G) = 1 + |G|\left(n - 1 + \sum_{i=1}^{\ell}\left(1 - \frac{1}{d_i}\right) + \sum_{j=n+1}^{m}\left(1 - \frac{1}{e_j}\right)\right).$$

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In Maclachlan's formula the last term is repeated. This is not true in any of McCullough's formulas. This small change makes a huge difference. It changes where the minimum value for the summation occurs.

For example, this small change means that we must consider abelian groups of all ranks when looking at the real genus spectrum.

Basic Results

It is easy to show that

$$\rho(Z_2 \times Z_2 \times Z_{2c}) = 1 + 4c.$$

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Let A be a finite abelian group of even order with $\rho(A)$ *positive. If* $\rho(A)$ *is not congruent to* $1 \pmod 4$*, then the Sylow* 2*-subgroup of A is cyclic.*

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Theorem

Let $g \geq 4$. If $g \in \mathcal{S}$ and $g-1$ is squarefree, then $g = \rho(A)$ for an abelian group A of rank two.

Main Results

We also obtain the following necessary condition for an integer g to be in the spectrum.

Theorem

Let $g \geq 4$. If $g \in \mathcal{S}$, then g satisfies one of the following conditions:

(i) $g \equiv 1 \pmod 4$*;*

(ii) $g \equiv 4 \pmod 6$ *or* $g \equiv 16 \pmod{20}$*;*

(iii) $g - 1$ *is divisible by* a *for some odd integer* a *such that* $(a-1)$ *divides* g*;*

(iv) $g - 1$ *is divisible by* p^2 *for some odd prime* p*;*

Density

It is a classical result that the squarefree integers have asymptotic density $6/\pi^2 \approx .6079$. The squarefree integers are distributed among the three classes of integers congruent to $1, 2, 3 \pmod 4$. Further, Jameson proved that the asymptotic density of the odd squarefree integers is $4/\pi^2 \approx .4053$. Using these results, we have the following bounds for the density.

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