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MINIMAL EXTENSION COVERS

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Let A and B be finite groups and let S be the set of all extensions of A by B. A group G is called an* extension cover *of* (A, B)*, if G contains all extensions in S as subgroups of G. A group G is called a* minimal extension cover *if G is an extension cover of minimal order. Let* $n = p_1^{e_1} \cdots p_k^{e_k}$ *be the prime factorization of the odd number n and define* $n_i = p_i^{e_i}$*. The group* $D_{n_1} \times \cdots \times D_{n_k} \times Z_2$ *is the unique minimal extension cover of* (Z_n, Z_2)*. This article also constructs a minimal extension cover of* (Z_{2^n}, Z_2)*. Some conjectures about minimal extension covers are examined as well.

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1. INTRODUCTION

Given finite groups A and B, let $\{G_i \mid 1 \leq i \leq n\}$ be the set of all groups, up to isomorphism, such that A is a normal subgroup of G_i and $B \cong G_i/A$. The set $\{G_i\}$ is the set of all extensions of A by B. A group G is called an *extension cover of* (A, B) if G contains each group of $\{G_i\}$ as a subgroup. Clearly, extension covers exist. The direct product $G_1 \times \cdots \times G_n$ is an extension cover of (A, B). It is also true that $A \wr B$, the wreath product of A by B, is an extension cover of (A, B). Finally, any group which contains an extension cover is also an extension cover.

It follows that the set of all extension covers forms a partially ordered set under inclusion. It is easy to show that there are many minimal elements under this ordering. The set of all extension covers may also be ordered by the order of the group. We will define a minimal extension cover of (A, B) as an extension cover with minimal order. Even under this definition, minimal extension covers are not unique, in general. However, the order of a minimal extension cover is unique. It is easy to find both upper and lower bounds for this order. However, it is much harder to find "good" upper and lower bounds for the order of a minimal extension cover.

There are also some obvious modifications of the idea of an extension cover. A group G is called a central extension cover of (A, B) if G contains each central extension of A by B as a subgroup. Given any groups (A, B), the direct product with amalgamation $G_1 \times_A \cdots \times_A G_n$ of all central extensions of A by B is a central extension cover of (A, B).

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2. NOTATION

For $n \geq 3$, let D_n be the group with generators X, Y and defining relations

$$X^n = Y^2 = (YX)^2 = 1. \qquad (1)$$

The group D_n is the dihedral group of order $2n$.

The nonabelian 2-groups that possess a maximal cyclic subgroup of index 2 are well-known groups. There are four families of 2-groups with this property, [3, §5.4]. These families can be constructed using the nontrivial automorphisms of a cyclic 2-group. For $n \geq 3$, the automorphism group is given by

$$Aut(Z_{2^n}) = \langle -1 \rangle \times \langle 5 \rangle \cong Z_2 \times Z_{2^{n-2}}. \qquad (2)$$

These power automorphisms are detailed in [8, p. 130]; also see [3, §§2.5, 5.4]. Next we describe these four families of 2-groups.

The first family is the family of dihedral groups where n is a power of 2. In order to make the notation consistent across the four families, we will define $D(m)$ to be the dihedral group of order 2^m. Therefore, $D(m) = D_{2^{m-1}}$.

For $m \geq 3$, let $DC(m)$ be the group with generators X, Y and defining relations

$$X^{2^{m-1}} = 1, \qquad X^{2^{m-2}} = Y^2, \qquad Y^{-1}XY = X^{-1}. \qquad (3)$$

The group $DC(m)$ is called the *dicyclic* group of order 2^m or sometimes the *generalized quaternion* group [3, p. 29].

For $m \geq 4$, let $QD(m)$ be the group with generators X, Y and defining relations

$$X^{2^{m-1}} = Y^2 = 1, \qquad YXY = X^{-1+2^{m-2}}. \qquad (4)$$

The group $QD(m)$ of order 2^m is called a *quasidihedral* group (or *semidihedral* group) [3, p. 191].

For $m \geq 4$, let $QA(m)$ be the group with generators X, Y and defining relations

$$X^{2^{m-1}} = Y^2 = 1, \qquad Y^{-1}XY = X^{1+2^{m-2}}. \qquad (5)$$

The group $QA(m)$ is a nonabelian group of order 2^m [3, p. 190]; we call these groups *quasiabelian* [6, p. 237].

Each group in these four families has a maximal cyclic subgroup, and, in fact, these four families of groups are characterized among all nonabelian 2-groups by this property [3, Th. 4.4, p. 193]. The three automorphisms of order 2 of the maximal cyclic group will be called inversion, the quasidihedral action, and the quasiabelian action. Inversion is also used to construct a dicyclic group, but the element of the group that gives rise to the inner automorphism which is inversion has order 4.

3. QUESTIONS

There are a great many questions associated with the idea of minimal extension covers. We will show in the next section that minimal extension covers are sometimes unique and sometimes not unique. Another question of interest is whether a minimal extension cover contains the completion of an amalgam of the groups involved.

An amalgam of two groups is a diagram $G \longleftarrow K \longrightarrow H$, where the arrows are embeddings. We may think of G and H as groups which intersect in a common subgroup, K. Thus any two extensions G and H of A by B contain a copy of A as a normal subgroup and are an amalgam of these groups. A completion of this amalgam is a group M which contains isomorphic copies of the groups G and H which intersect in a subgroup isomorphic to A. This common subgroup, A, must be embedded in G and H in the same way as the embeddings in the amalgam. We say that M contains the amalgam. This idea may be extended to more than just two groups with a common subgroup. Completions of amalgams of two finite groups always exist.

We may ask the following questions about minimal extension covers. If a completion of the amalgam of all of the extensions of A by B exists, must the smallest such completion be a minimal extension cover of A by B? The definition of a minimal extension cover says nothing about how the extensions are situated in the group, so this seems unlikely. Does a minimal extension cover need to contain any amalgam of the extensions?

It seems clear that the next step is to generate some examples of minimal extension covers. In the cases where A or B or both are members of some family of groups, it seems reasonable that at least one of the minimal extension covers will also occur as a member of a family of groups. The rest of this article will be concerned with constructing minimal extension covers for families of very elementary groups and proving that these extension covers are minimal.

4. ELEMENTARY EXAMPLES

We give a number of examples of extension covers. Suppose that A and B are both nonabelian finite simple groups. By the Schrier conjecture (which has been verified using the classification of finite simple groups), $Out(S)$ is a solvable group. Therefore, if A and B are nonisomorphic, then B acts trivially on A. In all such cases, the only extension of A by B is the direct product. So the minimal extension cover of A by B is the direct product.

Next we look at minimal extension covers of (Z_n, Z_2) where n is odd. If G is an extension of Z_n by Z_2, then G is a semidirect product. Now since $Aut(Z_p) \cong Z_{(p-1)p^{n-2}}$, it follows that Z_2 acts on Z_{p^n} either trivially or by inversion. The Primary Decomposition Theorem says that Z_n is isomorphic to the direct product of its p-primary factors and those factors are characteristic. Therefore, any automorphism of Z_n of order 2 acts either trivially or by inversion on the p-primary factors. So if there are k p-primary factors, then there are 2^k possible automorphisms of Z_n. Each one of these automorphisms corresponds to a non-isomorphic extension and vice versa. Therefore, there are 2^k extensions of order $2n$. Each of these extensions is isomorphic to the direct product of a dihedral group and a cyclic group, $D_u \times Z_v$, where u and v are relatively prime and $n = uv$.

Now let $n = p_1^{e_1} \cdots p_k^{e_k}$ be the prime factorization of n, and define $n_i = p_i^{e_i}$. Let $M(n) = D_{n_1} \times \cdots \times D_{n_k} \times Z_2$. Suppose that each dihedral group is generated by elements x_i and y_i of order 2 and n_i, respectively, and z generates Z_2. Given a group

$D_u \times Z_v$, where u and v are relatively prime, $u \neq 1$ and $n = uv$, let $u = m_1 \cdots m_s$ where the m_i are a rearrangement of the n_i. Therefore, $v = m_{s+1} \cdots m_k$. Define the elements x, y, and w by $x = x_{m_1} \cdots x_{m_s}$, $y = y_{m_1} \cdots y_{m_s}$, and $w = y_{m_{s+1}} \cdots y_{m_k}$. Clearly, x has order 2, y has order u, and x operates on y by inversion. Therefore, $\langle x, y\rangle \cong D_u$. Next, w has order v and commutes with both x and y. It follows that $\langle x, y, w\rangle \cong D_u \times Z_v$. Finally, $\langle z, y_1 \cdots y_k\rangle \cong Z_{2n}$ and $M(n)$ is an extension cover of (Z_n, Z_2).

It remains to prove that $M(n)$ is a minimal extension cover of (Z_n, Z_2). Since a computer search of all groups of order 30 up to and including order 120 shows that $M(15) = D_3 \times D_5 \times Z_2$ is the unique minimal extension cover of (Z_{15}, Z_2), it is clear that $M(n)$ is a good candidate for the minimal extension cover.

The remainder of this section will be devoted to proving that $M(n)$ is the unique minimal extension cover of (Z_n, Z_2).

Lemma 1. *Let G be a minimal extension cover of (Z_n, Z_2), where n is odd. Let $n = p^e \times m$ where p does not divide m. Suppose that a Sylow p-subgroup S of G has order p^e, then S is cyclic and normal in G. In addition, all elements of G act on S either trivially or as the inversion automorphism.*

Proof. Let G be a minimal extension cover of (Z_n, Z_2). Let S be any Sylow p-subgroup of G with p odd, and suppose that $|S| = p^e$ is the highest power of p dividing n. Let $H = D_u \times Z_v = Z_{uv} \times_\phi Z_2$ be any extension of Z_n by Z_2. Suppose Z_{uv} is generated by the element x. Define $m = n/(p^e)$. Now x^m has order p^e, and so it is contained in some Sylow p-subgroup T of G. There exists an element z of G such that $T^z = S$. Now H^z is a subgroup which is isomorphic to H and whose Sylow p-subgroup is S. It follows that H^z normalizes S. Therefore, every extension of Z_n by Z_2 is isomorphic to a subgroup of $N_G(S)$. Since G is a minimal extension cover, it must equal $N_G(S)$. Therefore, S is normal in G, and it is clearly cyclic.

Now define the reversing symmetry group of an element $a \in G$, $E_G(a) = \{x \in G | x^{-1}ax = a^{-1}\} \bigcup C_G(a)$. The reversing symmetry group is a group [2]. Let H be any extension of Z_n by Z_2, and let $S = \langle a\rangle$. So $H \leq E_G(a)$ for all extensions H and $E_G(a)$ is an extension cover. Since G is a minimal extension cover, it follows that $G = E_G(a)$.

Theorem 1. *Let $n = p_1^{e_1} \cdots p_k^{e_k}$ be the prime factorization of the odd number n and define $n_i = p_i^{e_i}$. The group $M(n) = D_{n_1} \times \cdots \times D_{n_k} \times Z_2$ is the unique minimal extension cover of (Z_n, Z_2).*

Proof. We have already shown that $M(n)$ is an extension cover of (Z_n, Z_2). The proof that it is minimal is by induction on k, the number of odd primes dividing the integer n. First, suppose that $k = 1$. There are two extensions of (Z_{p^e}, Z_2), namely, D_{p^e} and Z_{2p^e}. It is obvious that $D_{p^e} \times Z_2$ is the minimal extension cover.

Now suppose that the minimal extension cover of (Z_m, Z_2) is $M(m) = D_{n_1} \times \cdots \times D_{n_l} \times Z_2$, where l is the number of primes in m and $l < k$. Let G be a minimal extension cover of (Z_n, Z_2), where k is the number of primes dividing n. So $|G| \leq |M(n)| = n \cdot 2^{k+1}$. Let S_{p_i} be the Sylow p_i subgroup of G for $i = 1, \ldots, k$. Clearly, $Z_{p_i^{e_i}} \leq S_{p_i}$. Assume that $p_i^{e_i} < |S_{p_i}|$ for all i. Therefore, $n \cdot p_1 \cdots p_k \cdot |S_2| \leq n \cdot 2^{k+1}$, where S_2 is a Sylow 2-subgroup of G. Since $2 \leq |S_2|$, we see that $p_1 \cdots p_k \leq 2^k$, and this gives a contradiction. Therefore, there is some Sylow p-subgroup, S, whose order is the largest power of p dividing n. It follows from Lemma 1 that S is a cyclic normal Sylow p-subgroup of G and that every element of G acts either trivially or as the inversion automorphism on S.

Clearly, there are elements of G that do not centralize S. Therefore, $C_G(S)$ is a proper subgroup of G, and it follows that $|C_G(S)| \leq |G|/2 \leq n \cdot 2^k$. Let T be any Sylow q-subgroup of G. Now by Lemma 1, T acts trivially on S, as does S itself. Therefore, $C_G(S)$ contains all Sylow subgroups for any odd prime. Let $|S| = p^e$, and define $m = n/(p^e)$. It follows that m has $k-1$ primes dividing it. We will show that $C_G(S)/S$ is a minimal extension cover of (Z_m, Z_2) and use the induction hypothesis.

Let H be any extension of Z_m by Z_2. So $H = D_u \times Z_v$, where $m = uv$ and u and v are relatively prime. Define $K = D_u \times Z_{vp^e}$, an extension of Z_n by Z_2. Now $K \leq C_G(S)$ and, therefore, $H \leq C_G(S)/S$. So $C_G(S)/S$ is an extension cover of (Z_m, Z_2). It follows that $|C_G(S)/S| \leq m \cdot 2^k$ and so $C_G(S)/S$ is a minimal extension cover of (Z_m, Z_2). By the induction hypothesis, $C_G(S)/S \cong D_{n_1} \times \cdots \times D_{n_{k-1}} \times Z_2$. In particular, all odd Sylow p-subgroups of $C_G(S)/S$ are cyclic and normal. It is easy to show that this implies that all odd Sylow p-subgroups of G are cyclic and normal. This also shows that the Sylow 2-subgroup of $C_G(S)$ is elementary abelian. Therefore, we can conclude that G is the semidirect product of $N \cong Z_n$ by a Sylow 2-subgroup, S_2.

By the same argument as used in Lemma 1, each extension has an isomorphic copy with Z_2 part contained in a fixed Sylow 2-subgroup S_2. The normal subgroup N is isomorphic to $Z_{p_1^{e_1}} \times \cdots \times Z_{p_k^{e_k}}$, and let b_i generate each p_i-primary factor of N. Let x_i be the element of order 2 which acts as inversion on the p_i factor and acts trivially on the other factors. Therefore, $\langle b_i, x_i \rangle \cong D_{p_i^{e_i}}$. Finally, let y be the element of order 2 in S_2 which acts trivially on N. By minimality, $S_2 = \langle x_1, \ldots, x_k, y \rangle$, and it has order 2^{k+1}. Next, we show that S_2 is elementary abelian.

Suppose that $k = 2$. Since $\langle x_1, y \rangle \leq C_G(S_{p_2})$ and $\langle x_2, y \rangle \leq C_G(S_{p_1})$, the induction assumption shows that y is in the center of S_2. If x_1 and x_2 do not commute, then $x_1 \cdot x_2$ has order a power of two that is 4 or larger. In addition, $x_1 \cdot x_2$ acts on N by inverting all elements. However, since S_2 has order 8, it is easy to see that the extension $D_n \times Z_2$ is not contained in G. This contradiction shows that x_1 and x_2 commute and S_2 is elementary abelian.

In the case $k > 2$, each pair of elements x_i and x_j is contained in some centralizer $C_G(S_{p_l})$, and the induction assumption shows that they commute. Since y is contained in each centralizer, it commutes with everything. Therefore, S_2 is elementary abelian. Finally, $G = \langle b_1, x_1 \rangle \times \cdots \times \langle b_k, x_k \rangle \times \langle y \rangle \cong M(n)$ and G has the proper form. Therefore, the theorem is true by induction.

5. ELEMENTARY EXAMPLES WHICH ARE 2-GROUPS

Next we look at minimal extension covers of (Z_n, Z_2) where n is a power of 2. It is clear that a minimal extension cover of (Z_2, Z_2) is $(Z_2 \times Z_4)$. It is also easy to see that D_4, which is isomorphic to the wreath product $Z_2 \wr Z_2$, is a minimal extension cover of (Z_2, Z_2). Let G be the group defined by

$$G = \langle x, y, z \mid x^8 = y^4 = x^{-4}y^2 = y^{-1}xyx = z^{-2}y^2 = [x, z] = [y, z] = 1 \rangle. \quad (6)$$

The group G is the direct product of the dicyclic group of order 16 and the cyclic group of order 4 with amalgamated subgroup of order 2. The extensions of Z_4 by Z_2 are the quaternions, Q, the dihedral group, D_4, and the abelian groups, Z_8 and $Z_2 \times Z_4$. Clearly, $Q = \langle x^2, y \rangle$ and $Z_8 = \langle x \rangle$. It is easy to see that $D_4 = \langle x^2, yz \rangle$ and $Z_2 \times Z_4 = \langle y, yz \rangle$. Therefore, G is an extension cover of (Z_4, Z_2), and it has order 32.

Suppose that H is an extension cover of (Z_4, Z_2) with order 16. Therefore, Z_8 has index 2 in H, and hence it is normal. It follows that H is either an abelian, dihedral, dicyclic, quasidihedral, or a quasiabelian group. It is clear that H is not abelian, dihedral or quasiabelian, since none of them contains Q. The other two do not contain copies of $Z_2 \times Z_4$. Therefore, G is a minimal extension cover of (Z_4, Z_2).

However, this minimal extension cover is not unique. Using the computer algebra system, Magma, it is easy to show that the groups of order 32 with numbers 11, 38, 40, 42, 43, and 44 contain subgroups isomorphic to each of the groups Q, D_4, Z_8, and $Z_2 \times Z_4$. These are the only minimal extension covers of (Z_4, Z_2). The group G constructed above is group number 42. In at least two of these 6 groups, there are normal subgroups isomorphic to each of Q, D_4, Z_8, and $Z_2 \times Z_4$. It is also the case that in at least one of them, not all of the groups are normal subgroups. Finally, in the group G(32,11), none of the groups isomorphic to Q intersect any group isomorphic to Z_8 in a subgroup of order 4. This shows that a minimal extension cover need not contain an amalgam of the extensions of A by B over the subgroup A.

In the general case of (Z_{2^k}, Z_2) for $k \geq 3$, the extensions of Z_{2^k} by Z_2 are two abelian groups, $Z_{2^k} \times Z_2$ and $Z_{2^{k+1}}$, the dihedral group, $D(k+1)$, the dicyclic group, $DC(k+1)$, the quasidihedral group, $QD(k+1)$, and the quasiabelian group, $QA(k+1)$. It is not obvious what group is the minimal extension cover in this case. Therefore, we will examine the smallest case, $k = 3$. The extensions of Z_8 by Z_2 are abelian, Z_{16} or $Z_8 \times Z_2$, dihedral, $D(4)$, dicyclic, $DC(4)$, quasidihedral, $QD(4)$, or quasiabelian, $QA(4)$. Using Magma, it is easy to show that there is a unique group of order 64 that contains subgroups isomorphic to each of these groups. The minimal extension cover of (Z_8, Z_2) is $G = SmallGroup(64, 40)$. While this group contains all six extensions of Z_8 by Z_2, no more than three of them intersect in a copy of Z_8. The groups $Z_8 \times Z_2$, $D(4)$ and $DC(4)$ intersect in Z_8, and this group is normal in G. This amalgam generates a subgroup of G of order 32. The groups $Z_8 \times Z_2$, Z_{16}, and $QD(4)$, intersect in Z_8, and this group is normal in G. This amalgam generates all of G. There are no other amalgams of two or more of these groups in G.

6. MINIMAL EXTENSION COVERS OF $(Z_{2^{n-1}}, Z_2)$

In this section, we will find an extension cover of $(Z_{2^{n-1}}, Z_2)$ for $n \geq 4$. Let $S(n)$ be the set of relators $S(n) = \{u^{2^n}, v^{2^n}, [u, v], u^2v^{-2}\}$. Then the group $A(n) = \langle u, v \mid S(n) \rangle$ is an abelian group isomorphic to $(Z_{2^n} \times Z_2)$. Define the group $G(n)$ by the presentation

$$\langle u, v, x, y \mid S(n), y^2, [u, y], y^{-1}vyv^{-(1+2^{n-1})}, x^2, [x, y], x^{-1}uxu, x^{-1}vxv \rangle. \qquad (7)$$

Clearly, $A(n)$ is normal in $G(n)$, and x operates on it by inversion. So $H(n) = \langle u, v, x\rangle$ has order 2^{n+2} and it is normal in $G(n)$. Now conjugation by y fixes u and x and takes v into $v^{1+2^{n-1}}$. It is easy to verify that this is an automorphism of $H(n)$ of order 2, and so $G(n)$ is the semidirect product $H(n) \times_{\alpha} \langle y\rangle$ of order 2^{n+3}.

It is obvious that $\langle u\rangle \cong \langle v\rangle \cong Z_{2^n}$ and that $\langle u^2, y\rangle \cong Z_{2^{n-1}} \times Z_2$. Also $\langle u^2, x\rangle \cong D(n)$. Next, $(uv)^2 = u^4$, and so $|uv| = 2^{n-1}$ and $|y| = 2$. Also $y(uv)y = uv^{1+2^{n-1}} = (uv)^{1+2^{n-2}}$. It follows that $\langle uv, y\rangle \cong QA(n)$. Similarly, $(xy)(uv)(xy) = xuv^{1+2^{n-1}}x = u^{-1}v^{-1+2^{n-1}} = (uv)^{-1+2^{n-2}}$, and it follows that $\langle uv, xy\rangle \cong QD(n)$. Finally, $\langle v^2, xyv\rangle \cong DC(n)$. Hence $G(n)$ is an $(Z_{2^{n-1}}, Z_2)$ extension cover. By the previous section, the $n = 4$ case is NOT a minimal extension cover. In the $n = 5$ case, using Magma, it is possible to show that there is no (Z_{16}, Z_2) extension cover of order 128, and so $G(n)$ is a minimal extension cover.

Theorem 2. *The group $G(n)$ defined by presentation* (7) *is a minimal extension cover of $(Z_{2^{n-1}}, Z_2)$ for $n \geq 5$. In general, it is not unique.*

Proof. Now we prove that $G(n)$ is a minimal extension cover of $(Z_{2^{n-1}}, Z_2)$ for all $n \geq 5$. Suppose that $T(n)$ is a $(Z_{2^{n-1}}, Z_2)$ extension cover with $|T(n)| < 2^{n+3}$. Since 2^n divides $|T(n)|$, it follows that $|T(n)| = m \cdot 2^n$ with $1 < m < 8$. Since each $(Z_{2^{n-1}}, Z_2)$ extension is a 2-group, it is contained in a Sylow 2-subgroup of $T(n)$. Since all Sylow 2-subgroups are conjugate, we may replace $T(n)$ by its Sylow 2-subgroup. Therefore, we may assume that $m = 2$ or $m = 4$.

Suppose that $m = 2$. We have that Z_{2^n} is a normal subgroup of $T(n)$, and this implies that $T(n)$ is isomorphic to one of the groups $Z_{2^{n+1}}$, $Z_{2^n} \times Z_2$, $D(n+1)$, $DC(n+1)$, $QD(n+1)$, or $QA(n+1)$. None of these groups contains a subgroup isomorphic to $QA(n)$, and so we can eliminate this possibility.

Finally, suppose that $m = 4$. It follows that $T(n)$ contains a cyclic group of index 4. Clearly, $T(n)$ could contain a cyclic subgroup of index 2, in which case, it would be abelian, dihedral, dicyclic, quasidihedral, or quasiabelian. However, it is easy to show that none of these groups is an extension cover of $(Z_{2^{n-1}}, Z_2)$.

Therefore, we consider the families of 2-groups that have cyclic subgroups of index 4 but no cyclic subgroup of index 2. The classification of these groups is over a century old, accomplished by Burnside [1] and Miller [4, 5]. There are two abelian groups and 25 nonabelian groups of this type, as long as the order of the groups is 64 or larger. The fourteen groups studied by Burnside [1] all contain a cyclic normal subgroup of index 4 in the group. These groups are denoted by B_1 to B_{14}. In the eleven groups studied by Miller [4, 5], the cyclic group of index 4 is not normal. These groups are isomorphic to one of the groups in Table 1.

Suppose that the cyclic group has order 2^n and, therefore, the 25 non-abelian groups $G = T(n)$ have order 2^{n+2}. Each of these groups contains a subgroup of index 2 isomorphic to either $Z_2 \times Z_{2^n}$ or the quasiabelian group QA(n+1). This is true regardless of whether the cyclic group of index 4 is normal or not. Denote this subgroup by N.

Suppose first that N is abelian. Then we must have $N \cong Z_2 \times Z_{2^n}$, and the group G has a partial presentation

$$G = \langle x, y, z \mid x^{2^n} = y^2 = [x, y] = 1, \ldots\rangle. \tag{8}$$

Table 1 Nonabelian groups with cyclic subgroup of index 4

Family	Presentation	$z^{-1}xz =$	$z^{-1}yz =$	$z^2 =$
B_1	8	x	$yx^{2^{n-1}}$	1
B_2	8	x^{-1}	y	y
B_3	8	$x^{1+2^{n-1}}$	y	y
B_4	8	$x^{-1+2^{n-1}}$	y	y
B_5	10	$x^{1+2^{n-2}}$	y	y
B_6	10	$x^{-1+2^{n-2}}$	y	y
B_7	8	x^{-1}	y	$x^{2^{n-1}}$
B_8	8	$x^{1+2^{n-1}}$	y	1
B_9	8	$x^{-1+2^{n-1}}$	y	1
B_{10}	8	x^{-1}	y	1
B_{11}	8	x^{-1}	$yx^{2^{n-1}}$	1
B_{12}	8	x^{-1}	$yx^{2^{n-1}}$	$yx^{2^{n-2}}$
B_{13}	10	x^{-1}	y	1
B_{14}	10	$x^{-1+2^{n-1}}$	$yx^{2^{n-1}}$	1
M_1	8	xy	y	1
M_2	8	xy	y	y
M_3	8	$x^{-1+2^{n-2}}y$	$yx^{2^{n-1}}$	1
M_4	8	$x^{1+2^{n-2}}y$	$yx^{2^{n-1}}$	1
M_5	8	$x^{-1+2^{n-1}}y$	y	$x^{2^{n-1}}$
M_6	8	$x^{-1}y$	y	1
Q_1	10	xy	y	1
Q_2	10	$x^{-1}y$	$yx^{2^{n-1}}$	1
Q_3	10	$x^{1+2^{n-2}}y$	$yx^{2^{n-1}}$	1
Q_4	10	$x^{1+2^{n-2}}y$	y	y
Q_5	10	$x^{-1+2^{n-2}}y$	y	1
Q_6	10	$x^{-1+2^{n-2}}y$	y	$x^{2^{n-1}}$

Ten of Burnside's 14 groups are of this type. A complete presentation for G would specify the action of a third element z, outside of N, on x and y.

In each of the remaining four of Burnside's groups, the subgroup N of index 2 is quasi-abelian with presentation

$$x^{2^n} = y^2 = 1, \qquad y^{-1}xy = x^{1+2^{n-1}}. \tag{9}$$

In this case, the group G has a partial presentation

$$G = \langle x, y, z \mid x^{2^n} = y^2 = 1,\ y^{-1}xy = x^{1+2^{n-1}}, \ldots \rangle. \tag{10}$$

Now assume that the nonabelian 2-group G has cyclic subgroups of index 4, but that none of these is normal. Again, we assume that G does not have a cyclic subgroup of index 2. Let $H = \langle x \rangle$ be one of the big cyclic subgroups. Then, if the order is 64 or larger, there are exactly 11 nonisomorphic groups.

In this case, let N be the normalizer of H in G. Then N is of order 2^{n+1} and $H \subset N \subset G$. Again, the two possibilities are that N is abelian or quasi-abelian. This follows because the normalizer must contain two conjugates of H, and the dihedral, quasidihedral, and dicyclic groups have a unique cyclic subgroup of index 2.

Suppose first that $N \cong Z_2 \times Z_{2^n}$. Six of Miller's groups are of this type [4]. We denote these groups $M_i(n+2)$ for $1 \leq i \leq 6$. Suppose finally that $N = N(H)$ is quasi-abelian with presentation (9). There are also six groups of this type [5], which we denote $Q_i(n+2)$ for $1 \leq i \leq 6$. The group M_4 and the group Q_3 are isomorphic, which gives 11 nonisomorphic groups. Table 1 contains these 26 groups for $n \geq 4$. If n is 2 or 3, then there are some isomorphisms between groups, and fewer distinct groups are defined.

The subgroup N has two cyclic subgroups of order 2^n as well as two cyclic subgroups of order 2^{n-1}. Let A be either of the cyclic groups of order 2^{n-1} in N. It is easy to check that $N \leq C_G(A)$. In the groups B_2, B_4, B_6, B_7, B_9, B_{10}, B_{11}, B_{12}, B_{13}, B_{14}, M_3, M_5, M_6, Q_2, Q_5, and Q_6, these two cyclic subgroups of order 2^{n-1} in N are the only subgroups of this order in the full group. The elements of the nontrivial coset of N act by an automorphism on both subgroups of order 2^{n-1}. Usually, but not always, the automorphism is the same on both groups. Since the dihedral and dicyclic groups both share the inversion automorphism, this means that there can be at most three different nonabelian extensions of $Z_{2^{n-1}}$ by Z_2. Therefore, these 16 groups are not extension covers of $(Z_{2^{n-1}}, Z_2)$.

The remaining nine groups contain four distinct cyclic groups of order 2^{n-1}. We begin by giving the presentation for the group $B_1(n+2)$ of order 2^{n+2}:

$$B_1 = \langle x, y, z \mid x^{2^n} = y^2 = z^2 = [x, y] = [x, z] = 1, z^{-1}yz = yx^{2^{n-1}} \rangle. \tag{11}$$

By checking the orders of each element in this group, we can tell that this group has four subgroups of order 2^{n-1}. These groups are $\langle x^2 \rangle$, $\langle x^2 y \rangle$, $\langle x^2 z \rangle$, and $\langle x^2 yz \rangle$. The subgroup N described above is $\langle x, y \rangle$. Let us consider the case of the subgroup $\langle x^2 yz \rangle$. It is easy to compute that $C_G(\langle x^2 yz \rangle) = \langle x, yz \rangle$ and that $N_G(\langle x^2 yz \rangle) = G$. So the element z has order 2 and conjugation by z is the quasiabelian action on $\langle x^2 yz \rangle$. It follows that the quasiabelian group $QA(n)$ is an extension of $Z_{2^{n-1}}$ by Z_2 contained in G. The other three cyclic subgroups yield either the quasiabelian group also or an abelian group. It follows that $B_1(n+2)$ is not an extension cover of $(Z_{2^{n-1}}, Z_2)$.

The groups $B_3(n+2)$ and $B_8(n+2)$ each contains four cyclic subgroups of order 2^{n-1} which are normal in G, and the automorphisms of these groups defined by conjugation is either quasiabelian or trivial. In this case, there is no element of order two where conjugation by that element is the quasiabelian automorphism, and so all of the extensions of $Z_{2^{n-1}}$ by Z_2 contained in $B_3(n+2)$ and $B_8(n+2)$ are abelian. It follows that $B_3(n+2)$ and $B_8(n+2)$ are not extension covers of $(Z_{2^{n-1}}, Z_2)$.

The group $B_5(n+2)$ contains four cyclic subgroups of order 2^{n-1} which are normal in G. For two of these groups, the index of the centralizer in G is 2, and the automorphism is the quasiabelian automorphism. Since these automorphisms are given by conjugation by elements of order greater than 2, the only extensions of these groups are abelian groups. For the other two cyclic groups, the group $G/C_G(H)$, where H is the cyclic subgroup, is cyclic of order 4 and it is generated by an automorphism of H of order 4. Again, the only automorphisms of order 2 are quasiabelian. Again these automorphisms are given by conjugation by elements of order greater than 2 and so the only extensions of these groups are abelian groups. It follows that $B_5(n+2)$ is not an extension cover of $(Z_{2^{n-1}}, Z_2)$.

The groups $M_1(n+2)$ and $M_2(n+2)$ each contains four cyclic subgroups of order 2^{n-1}. For two of these subgroups, the centralizer is the entire group. For the other two, the centralizer and the normalizer are equal and have index 2 in the group. Thus, all extensions in G are abelian, and they are not extension covers of $(Z_{2^{n-1}}, Z_2)$. The group $Q_4(n+2)$ is similar, except that one of the two normal subgroups has centralizer of index 2 in G with a quasiabelian action which is not realized by any elements of order 2.

The groups $Q_1(n+2)$ and $Q_3(n+2)$ each contains four cyclic subgroups of order 2^{n-1}. Two of these groups are normal in the whole group. In $Q_1(n+2)$, both groups have centralizer of index 2. In $Q_3(n+2)$, one of these groups has centralizer of index 2 and, in the other, the centralizer is the whole group. In the three nontrivial cases above, the action is the quasiabelian action. In both groups, $Q_1(n+2)$ and $Q_3(n+2)$, the remaining two cyclic subgroups have normalizer of index 2 in the whole group and centralizer of index 2 in the normalizer. In each of these cases, the action is the quasiabelian action. Therefore, these two groups are not extension covers of $(Z_{2^{n-1}}, Z_2)$.

It follows that if $T(n)$ is a group with $|T(n)| < 2^{n+3}$, then $T(n)$ is not a $(Z_{2^{n-1}}, Z_2)$ extension cover. Therefore, $G(n)$ is a minimal extension cover.

In the $n = 5$ case, using Magma, it is possible to show that there are 12 nonisomorphic (Z_{16}, Z_2) extension covers of order 256, and so $G(5)$ is not a unique minimal extension cover. In fact, $G(5) \cong SmallGroup(256, 26970)$ in the Magma Library.

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