THE SYMMETRIC GENUS OF LARGE ODD ORDER GROUPS

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Abstract. *Let G be a finite group of odd order. The symmetric genus $\sigma(G)$ is the minimum genus of any Riemann surface on which G acts faithfully. Suppose G acts on a Riemann surface X of genus $g \geq 2$. If $|G| > 8(g-1)$, then $|G| = K(g-1)$, where K is 15, $\frac{21}{2}$, 9 or $\frac{33}{4}$. We call these four types of groups LO-1 groups through LO-4 groups, respectively. These groups are quotients of Fuchsian triangle groups of type $(3,3,n)$, for $n = 5,7,9$ and 11, respectively. They are generated by two elements of order 3, and we obtain restrictions on the powers of the primes dividing the orders of such groups. Let J be the set of integers g for which there is an LO-group of symmetric genus g. These restrictions are sufficient to prove that the set J has density 0 in the set of positive integers. In addition, we study the metabelian LO-3 groups, and classify the integers that are the genus values of metabelian LO-3 groups.*

1. Introduction.

Let G be a finite group. Among the various genus parameters associated with the group G, the most classical is perhaps the *strong symmetric* genus $\sigma^0(G)$, the minimum genus of any Riemann surface on which G acts faithfully and preserving orientation. Work with this parameter dates back over a century and includes the fundamental $84(g-1)$ bound of Hurwitz [5].

A closely related parameter is the *symmetric* genus $\sigma(G)$, the minimum genus of any Riemann surface on which G acts (possibly reversing orientation). Obviously $\sigma(G) \leq \sigma^0(G)$ always, but in some (important) cases, the two parameters agree. If the group G does not have a subgroup of index 2, then G cannot act on a surface reversing orientation and thus $\sigma(G) = \sigma^0(G)$. In particular, if G is a group of odd order, then $\sigma(G) = \sigma^0(G)$.

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Here we consider groups of odd order, in particular, the groups of odd order that act as "large" automorphism groups of odd order of Riemann surfaces. If the odd order group G acts on a Riemann surface of genus $g \geq 2$, then $|G| \leq 15(g-1)$. Further, if $|G| > 8(g-1)$, then $|G| = K(g-1)$, where K is 15, $\frac{21}{2}$, 9 or $\frac{33}{4}$. We call these four types of groups *LO-1 groups* through *LO-4 groups*, respectively, and an *LO-group* means a group of one of these four types. These groups are quotients of Fuchsian triangle groups of type $(3,3,n)$, for $n = 5,7,9$ and 11, respectively. Also, three of the four are quotients of triangle groups of type $(3,3,p)$, where p is a prime. Not surprisingly, whether the prime p is congruent to 1 or 2 modulo 3 is important. The types of groups considered as LO-groups could be expanded by allowing the non-euclidean area of the triangle group to be larger and the lower bound $8(g-1)$ to be smaller. With a larger non-euclidean area for the triangle group, not all of the quotient groups would be generated by two elements of order 3. Also the bound $8(g-1)$ is a nice one, numerically. Therefore, these LO-groups are our main focus. We consider some general theorems about solvable groups generated by two elements of order 3 in Section 3.

The basic results about the LO-groups were obtained in [8]; the emphasis there was on LO-1 groups with some consideration of LO-2 groups. We are interested in the positive integers that are the symmetric genus of an LO-group. While there are infinitely many LO-1 groups (of infinitely many different genera) [8, Cor. 5], these groups occur rather infrequently. For example, there are LO-1 groups of order 75 and 375, but the "next" LO-1 group that we know has order 9375. The situation for LO-2 and LO-4 groups is similar. Of course, LO-3 groups are different, and there is a richer supply of these groups, but, still, the integers that are the genus values of LO-3 groups are quite limited.

To interpret our results we use the standard notion of density. One of our main results is the following.

Theorem 1. *Let J be the set of integers g for which there is a $LO-$ group of symmetric genus g. Then the set J has density 0 in the set of positive integers.*

To establish this result, we consider the integers that are the orders of each of the types of LO-groups. In each case, we obtain a restriction on the powers of the primes dividing the group order. For example, let G be an LO-2 group. Then if q is a prime, $q > 3$, $q \neq 7$, such that q divides $|G|$, then we show that q^2 divides $|G|$. We can do better than this for *LO*-1 and *LO*-4 groups, and *LO*-3 groups have similar results. These restrictions are sufficient to prove Theorem 1.

The techniques employed can be generalized to give results about solvable groups generated by elements of order 3. For instance, we have the following interesting result.

Theorem 2. *Let L be the set of integers n for which there is a solvable group G generated by two elements of order 3 such that $|G| = n$. Then the set L has density 0 in the set of positive integers.*

In addition, we study the metabelian *LO*-3 groups. We construct a family of finite groups so that every metabelian *LO*-3 group is a quotient of one of the groups in the family. We then use this family to classify the integers that are the genus values of metabelian *LO*-3 groups.

2. Background Results.

We use the standard well-known approach to group actions on surfaces of genus $g \geq 2$. Let the finite group G act on the (compact) Riemann surface X of genus $g \geq 2$. Represent $X = \mathcal{H}/K$, where $\mathcal{H}$ is the hyperbolic plane and K is a Fuchsian surface group. Then obtain a Fuchsian group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that $K = kernel\ \phi$. Associated with the Fuchsian group Γ are its signature and canonical presentation. It is basic that each period of Γ divides $|G|$. Further, the non-euclidean area $\mu(\Gamma)$ of a fundamental region for Γ can be calculated directly from its signature [11, p.235]. Then the genus of the surface X on which G acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/4\pi. \tag{1}$$

Especially important in the study of large group actions on Riemann surfaces are the triangle groups. A Fuchsian group is a triangle group if it has signature

$$(0; +; [\ell, m, n]; \{\}), \text{ where } 1/\ell + 1/m + 1/n < 1.$$

We denote a group with this signature by $\Gamma(\ell, m, n)$.

The general upper bound 168(g - 1) for the size of the (full) automorphism group of a surface of genus $g \geq 2$ can be improved considerably for groups of odd order. The following result [8, Prop. 1] is elementary.

Proposition 1. *Let G be a finite group of odd order. Suppose G acts on a Riemann surface X of genus $g \geq 2$. Then $|G| \leq 15(g-1)$. Further, if $|G| > 8(g-1)$, then $|G|$ is one of the following; in each case G is a quotient of the listed triangle group by a surface group.*

1) $\lvert G\rvert = 15(g-1)$	$\Gamma(3,3,5)$
2) $\lvert G\rvert = \frac{21}{2}(g-1)$	$\Gamma(3,3,7)$
3) $\lvert G\rvert = 9(g-1)$	$\Gamma(3,3,9)$
4) $\lvert G\rvert = \frac{33}{4}(g-1)$	$\Gamma(3,3,11)$

Thus the group G is an *LO*-group if and only if G is a quotient Δ/K, where Δ is the triangle group $\Gamma(3,3,n)$ with the period $n = 5$, 7, 9 or 11 and K is a surface group. The group Δ has presentation

$$x^3 = y^3 = (xy)^n = 1. \quad (2)$$

Since the surface group K contains no elements of finite order, the quotient G is generated by two elements X, Y of order 3 such that the product XY has order n. Hence G has partial presentation (2), with the relation $(xy)^n = 1$ fulfilled. If n is not a power of 3, then G is solvable, but not nilpotent.

The case $n = 9$ is different, of course. The *LO*-3 group G can be a 3-group and hence nilpotent. However $|G|$ can also involve primes other than 3, in which case, it is solvable, but not nilpotent. In any case, an *LO*-3 group must have a nilpotent *LO*-3 quotient group. In general, these groups require different ideas, and much of our work here concerns *LO*-3 groups.

The only odd order groups that have (strong) symmetric genus 0 are cyclic [4, Th. 6.3.1, p. 285]. Clearly, just from partial presentation (2), no *LO*-group is cyclic and has genus zero. However, there are *LO*-groups of genus 1. Obviously, no *LO*-group can be abelian. It follows that an *LO*-group with $\sigma = 1$ must be in class (c) [4, p. 281]. A group in class (c) has partial presentation

$$x^3 = y^3 = (xy)^3 = 1. \quad (3)$$

Next we present a family of groups of odd order that contain some small *LO*-groups. Let p be an odd prime, $p \geq 5$, and let $Z_p \times Z_p$ have presentation

$$X^p = Y^p = 1, XY = YX. \quad (4)$$

Then let the group H_{3p^2} be the nonabelian group of order $3p^2$ with generators X, Y and A and relations (4) together with

$$A^3 = 1, A^{-1}XA = Y, A^{-1}YA = X^{-1}Y^{-1}. \quad (5)$$

The following basic result was established in [8, Prop. 4].

Proposition 2. *Let p be an odd prime, $p \geq 5$. Then*
(1) $\sigma(H_{3p^2}) = 1$.
(2) H_{3p^2} is a quotient of a triangle group $\Gamma(3,3,p)$ with kernel a surface group, and H_{3p^2} acts on a surface of genus $1 + p(p-3)/2$.

The group $H_{3\cdot 5^2}$ is an LO-1 group with an action on a surface of genus 6. We denote this group H_{75}. The groups $H_{3\cdot 7^2}$ and $H_{3\cdot 11^2}$ are LO-groups as well.

The derived series of LO-groups were also considered in [8], with the exception of LO-3 groups. The following is basic; for a proof, see [8, Lemma 1].

Lemma 1. *Let p be an odd prime, $p \geq 5$, and let $\Gamma = \Gamma(3,3,p)$. Then*
(1) $\Gamma/\Gamma' \cong Z_3$
(2) $\Gamma'/\Gamma'' \cong Z_p \times Z_p$
(3) $\Gamma/\Gamma'' \cong H_{3p^2}$

Next we consider the first two steps of the derived series of a quotient of $\Gamma(3,3,p)$ by a surface group. There are two cases, depending upon whether p is congruent to 1 or 2 modulo 3.

Proposition 3. *Let p be a prime congruent to $2\ (mod\ 3)$, and let G be a quotient of $\Gamma(3,3,p)$ by a surface group. Then*
(1) $G/G' \cong Z_3$
(2) $G'/G'' \cong Z_p \times Z_p$
(3) $G/G'' \cong H_{3p^2}$

Proof: By Lemma 1, G/G'' is the image of H_{3p^2}. If the kernel of this map K is non-trivial, then G/K is cyclic, since an element of order 3 cannot act non-trivially on a group of order p.

Next assume $p \equiv 1\ (mod\ 3)$. Then the single nonabelian group of order $3p$ has presentation

$$X^p = Y^3 = 1, Y^{-1}XY = X^r, \tag{6}$$

where $r^3 \equiv 1\ (mod\ p)$ and $r \not\equiv 1\ (mod\ p)$. We denote this group G_{3p}. The following result was established in [8, Prop. 7].

Proposition 4. *Let p be an odd prime such that $p \equiv 1\ (mod\ 3)$. Then*
(1) $\sigma(G_{3p}) = 1$.
(2) G_{3p} is a quotient of a triangle group $\Gamma(3,3,p)$ with kernel a surface group and G acts on a surface of genus $1 + (p-3)/2$.

In particular, the group $G_{3\cdot 7}$ is an LO-2 group that acts on a surface of genus 3; henceforth we denote this group G_{21}.

Continue to assume that p is an odd prime such that $p \equiv 1\ (mod\ 3)$. In this case the group H_{3p^2} has a normal subgroup N of order p such that $H_{3p^2}/N \cong G_{3p}$; for more details, see [8, Prop. 8]. Now we have the analog of Proposition 3 for groups of this type.

Proposition 5. *Let p be a prime congruent to* 1 (*mod* 3)*, and let G be a quotient of* $\Gamma(3, 3, p)$ *by a surface group. Then* $G/G' \cong Z_3$*. Further,* G/G'' *is isomorphic to either* G_{3p} *or* H_{3p^2}*; in either case, G has a normal subgroup M such that* $G/M \cong G_{3p}$*.*

Since the derived length of any group in class (c) is at most two [4, Th.6.3.4, p.296], these results yield a classification of groups of this type with symmetric genus one.

Theorem 3. *Let G be a quotient of* $\Gamma(3, 3, p)$ *by a surface group.*
(1) Let p be a prime congruent to 2 (*mod* 3)*. Then* $\sigma(G) = 1$ *if and only if* $G \cong H_{3p^2}$*.*
(2) Let p be a prime congruent to 1 (*mod* 3)*. Then* $\sigma(G) = 1$ *if and only if G is isomorphic to* G_{3p} *or* H_{3p^2}*.*

Some of our main results use the standard notion of density. Let I be a set of positive integers. For a positive integer n, let $f(n)$ denote the number of integers in the set I that are less than or equal to n. Then the natural density $\delta(I)$ of I in the set of positive integers is

$$\delta(I) = \lim_{n\to\infty} \frac{f(n)}{n}.$$

In stating results, it is sometimes convenient to use the standard concept of asymptotic functions. The function $f(x)$ is said to be *asymptotic* to the function $g(x)$, written $f(x) \sim g(x)$, if and only if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

3. Solvable groups generated by elements of order 3.

Now we establish two results about solvable groups generated by elements of the same prime order. Both results apply to LO-groups.

Theorem 4. *Let G be a finite solvable group which is generated by two or more elements of order p, where p is an odd prime, and let q be a prime not congruent to* 1 (*mod* p)*. If q divides* $|G|$*, then* q^2 *divides* $|G|$*.*

Proof: Suppose to the contrary that $|G| = qm$, where q does not divide m. Consider the factors $G^{(n-1)}/G^{(n)}$ in the derived series. There exists a value of $n \geq 2$, so that q divides the order of the factor $G^{(n-1)}/G^{(n)}$. Now $G^{(n-1)}/G^{(n)} \cong Z_p \times M$, where q does not divide $|M|$. Therefore, if Q is the Sylow q-subgroup of $G/G^{(n)}$, then Q is normal in $G/G^{(n)}$. By the Schur-Zassenhaus Theorem, $G/G^{(n)} \cong Q \times_\phi H$, where H is a subgroup of $G/G^{(n)}$. Now $H/C_H(Q)$ is isomorphic to a subgroup of the cyclic group $Aut(Q)$ of order $q - 1$. It follows that $H' \subseteq C_H(Q)$.

Since H is generated by elements of order p, H/H' is an elementary abelian p-group. It follows that either $H = C_H(Q)$ or $|H/C_H(Q)| = p$. If $H = C_H(Q)$, then $G/G^{(n)} \cong Q \times H$ and G cannot be generated by elements of order p. It follows that $|H/C_H(Q)| = p$ and p divides $q-1$, which is also a contradiction. Therefore, q^2 divides $|G|$.

Corollary 1. *Let G be a finite solvable group which is generated by two or more elements of order 3, and let q be a prime congruent to* 2 (*mod* 3)*. If q divides $|G|$, then q^2 divides $|G|$.*

We specialize to quotients of $\Gamma(3,3,p)$ and obtain a stronger result.

Theorem 5. *Let $p \neq 3$ be an odd prime, and let G be an odd order group that is the quotient of $\Gamma(3,3,p)$ by a surface group. Let q be an odd prime other than* 3 *or p. If q divides $|G|$, then q^2 divides $|G|$. Furthermore, in case p is congruent to* 2 (*mod* 3)*, if q divides $|G|$, then q^3 divides $|G|$.*

Proof. Assume that the odd prime q divides $|G|$. Among all quotients of G that are the quotient of $\Gamma(3,3,p)$ by a surface group and also have order divisible by q, let J be one of smallest order. Let M be a minimal normal subgroup of J contained in J''. Since G is solvable, the subgroup M is an elementary abelian π-group for some prime π. But by the minimality of J, the prime π must equal q and the index $[J:M]$ must be relatively prime to q, since otherwise the quotient J/M would be an image of $\Gamma(3,3,p)$ with order divisible by q and smaller than $|J|$. Hence we can write $M \cong (Z_q)^r$ for some positive integer r.

Let $H = J/M$. Then H is a quotient of $\Gamma(3,3,p)$ by a surface group, because $|M|$ is relatively prime to 3 and p. Since q does not divide $|H|$, we know that the group $J \cong M \times_\phi H$, where $\phi : H \longrightarrow Aut(M)$, by the Schur-Zassenhaus Theorem. Define $K = \text{kernel}(\phi)$. We show that $[H:K] > 3$. First, $K \neq H$, since $M \times K$ is obviously not a quotient of $\Gamma(3,3,p)$.

Suppose that $[H:K] = 3$. Now J has a normal subgroup $L \cong M \times K$ and $[J:L] = [H:K] = 3$. Hence, by Proposition 3, $L = J'$. But q divides $|L/L'|$, since M is abelian. This is a contradiction, since $L/L' \cong J'/J''$ and J is an image of $\Gamma(3,3,p)$ so that Proposition 3 applies. Therefore, $[H:K] > 3$.

Suppose that $r = 1$ and hence $M \cong Z_q$. Now the quotient group H/K is an image of $\Gamma(3,3,p)$. Let $H = \langle a, b\rangle$, where $|a| = |b| = 3$ and $|ab| = p$. Now, if the image of AB has order 1 in H/K, then $[H:K] = 3$, since $H/K = \langle aK \rangle$.

Further, H/K is isomorphic to a subgroup of $Aut(M) = Z_{q-1}$. Therefore, H/K is abelian and we get a contradiction. Clearly, $r \neq 1$.

Suppose that $r = 2$ and p is congruent to 2 (mod 3). Then $H/K \subseteq Aut(M) = GL(2,q)$ and the group H/K has a non-trivial Sylow p-subgroup P. The order of $GL(2,q)$ forces the prime q to be congruent to ± 1 (mod p). Also, $P \subseteq H'K/K \subseteq SL(2,q)$, since $SL(2,q)$ is the commutator subgroup of $GL(2,q)$.

By Proposition 3, P cannot be cyclic. This is a contradiction, because the Sylow p-subgroups of $SL(2,q)$ are cyclic [3, p. 42]. Thus $r \geq 3$, and q^3 divides $|G|$.

Now we consider quotients of the triangle group $\Gamma(3,3,3n)$. It is possible that G is a 3-group if n is a power of 3. The 3-groups that are quotients of $\Gamma(3,3,9)$ were studied by Zomorrodian [12]. Assume first that G is a 3-group that is an image of $\Gamma(3,3,3^k)$ for $k > 1$ by a surface group. Then 27 divides $|G|$, obviously. It is not hard to see, in an elementary way, that $|G|$ cannot be 27; for details, see [12, p. 240]. Thus, if the LO-3 group G is a 3-group, $|G| \geq 81$.

The classification of the first few terms of the lower central series of the triangle group $\Gamma(3,3,9)$ by Zommorodian [12, p. 242] may be extended to quotients of the triangle group $\Gamma(3,3,3n)$.

Lemma 2. *Let* $\Gamma = \Gamma(3,3,3n)$. *Then*
(1) $\Gamma/\Gamma' \cong Z_3 \times Z_3$.
(2) $\Gamma'/\Gamma'' \cong Z_n \times Z_n \times Z \times Z$.

Proof: Let $\Gamma = \Gamma(3,3,3n)$ have presentation $\langle x, y | x^3 = y^3 = (xy)^{3n} \rangle$. Define $u = [y^{-1}, x^{-1}]$, $v = [x, y^{-1}]$, $w = [y, x^{-1}]$ and $z = w^{x^{-1}}$. It is easy to show that

$$u^x = v, v^x = u^{-1}v^{-1}, w^x = z^{-1}w^{-1}, z^x = w \tag{7}$$

and

$$u^y = w^{-1}, v^y = wz, w^y = uw^{-1}, z^y = wu^{-1}v^{-1}z^{-1}w^{-1}. \tag{8}$$

Therefore $\Gamma' = \langle u, v, w, z \rangle$. Also, the group Γ' has relators

$$(vw)^n = (uz)^n = (uwzv)^n = 1.$$

Up to this point, all of the calculations have been in the group Γ. For the rest of the proof, we will consider the elements u, v, w and z as representing cosets in Γ'/Γ''.

It follows that

$$(uz)^x = vw, (vw)^x = (uz)^y = (uz)^{-1}(vw)^{-1} \textit{ and } (vw)^y = uz. \tag{9}$$

Therefore, $N = \langle uz, vw \rangle \cong Z_n \times Z_n$ is a normal subgroup of Γ/Γ''. It is also known that since Γ'/Γ'' is finitely generated, it is a direct product of finite and infinite cyclic groups. This completes the proof.

In the rest of the paper, we will consider LO-3 groups, instead of general quotients of $\Gamma = \Gamma(3, 3, 3n)$. The techniques applied will work in the general case, however.

4. LO-1, LO-2 and LO-4 groups.

Our first goal is to consider the set of integers g for which there is a LO-1 group of symmetric genus g and show that this set has density zero in the set of positive integers. The key to the proof is the following application of Theorem 5.

Corollary 2. *Let G be an LO-1 group and let q be a prime, $q > 5$. If q divides $|G|$, then q^3 divides $|G|$.*

This is the best possible result of this type, since it is possible to construct a semi-direct product $(Z_{11})^3 \times_\phi H_{75}$ that is an LO-1 group of order $99825 = 3 \cdot 5^2 \cdot 11^3$.

The power of 3 in the order of an LO-1 group is restricted.

Proposition 6. *Let G be an LO-1 group. If 9 divides $|G|$, then $81 = 3^4$ divides $|G|$.*

Proof. Let G be an LO-1 group such that 9 divides $|G|$. By Proposition 3, we know that $[G : G''] = 75$ so that 3 divides $|G''|$. Let P be the Sylow 3-subgroup of G'' Assume that $|P|$ is 3 or 9. Then in either case, P has a normal 3-complement Q in G''; this is a consequence of Burnside's Theorem [10, pp. 137, 138, 141]. But Q is characteristic in G'' and hence normal in G. Then the quotient group G/Q is an LO-1 group of order $225 = 75 \cdot 3$ or $675 = 75 \cdot 9$. There is no LO-1 group of either order. Hence $|P| \geq 27$ and 81 divides the order of G.

The order of an LO-1 group is always divisible by 25, of course. The order of the LO-1 group of order 375 is divisible by 5^3, but we do not know if a result like Proposition 6 about powers of 5 is possible.

Let G be an LO-1 group. Then by Corollary 2 and Proposition 6, the order of G has the form

$$|G| = 3^i \cdot 5^j \cdot p_1^{n_1} \cdot p_2^{n_2} \cdots p_t^{n_t}, \tag{10}$$

where $i = 1$ or $i \geq 4$, $j \geq 2$, each p_i is a prime larger than 5, and each exponent $n_i \geq 3$ (possibly $t = 0$).

Now we turn our attention to the general problem of classifying the integers g for which there is a LO-group of symmetric genus g. Let J_1 be the set of integers g for which there is a LO-1 group of symmetric genus g. We know that the set J_1 is an infinite set, since there is a construction that shows that there are infinitely many extensions of abelian groups by *LO*-1 groups that are also *LO*-1 groups. The construction uses the theory of covering spaces and the fundamental group; see [8, Section 4]. This construction only shows the existence of infinite families of *LO*-1 groups, however, and does not produce presentations of the groups. Corollary 2 and a result from analytic number theory combine to yield the density of J_1.

A positive integer n is said to be *cube-full* if whenever the prime p divides n, p^3 divides n. The density of the set of cube-full integers is well-understood; see [6, Section 14.4].

Lemma 3. *Let $f(n)$ be the number of cube-full integers less than or equal to n. Then*

$$f(n) \sim Cn^{1/3},$$

where C is a constant.

Let G be an *LO*-1 group of symmetric genus $g = \sigma(G)$. Then by Proposition 1, $g = 1 + |G|/15$, and thus from (7) $g = 1 + M/25$, where M is a cube-full integer. This gives the following.

Theorem 6. *Let J_1 be the set of integers g for which there is a LO-1 group of symmetric genus g. Then the set J_1 has density zero in the set of positive integers.*

Next we briefly consider the other *LO*-groups that are similar to *LO*-1 groups as quotients of triangle groups $\Gamma(3, 3, p)$, where p is a prime. First *LO*-4 groups are quotients of the triangle group $\Gamma(3, 3, 11)$ and 11, like 5, is congruent to 2 (*mod* 3). We note the following application of Theorem 5.

Corollary 3. *Let G be an LO-4 group and let q be a prime, $q > 3$, $q \neq 11$. If q divides $|G|$, then q^3 divides $|G|$.*

The key point is that the Sylow 11-subgroups of $SL(2, q)$ are cyclic, where the prime q must be congruent to ± 1 (*mod* 11) [3, p. 42]. The analog of Theorem 6 holds in the same way for *LO*-4 groups.

Theorem 7. *Let J_4 be the set of integers g for which there is a LO-4 group of symmetric genus g. Then the set J_4 has density zero in the set of positive integers.*

For *LO*-2 groups, Theorem 5 gives the following.

Corollary 4. *Let G be an LO-2 group and let q be a prime, $q > 3$, $q \neq 7$. If q divides $|G|$, then q^2 divides $|G|$.*

There exist *LO*-2 groups with a cyclic Sylow 7-subgroup, however, so that the final part of the proof of Theorem 5 cannot be used. It is possible, for example, to construct an *LO*-2 group of order $27951 = 3 \cdot 7 \cdot 11^3$. However, we do not know an example of an *LO*-2 group G such that $|G|$ is "only" divisible by the square of a prime other than 3 and 7.

Nevertheless, Corollary 4 is enough to determine the analog of Theorem 6 for *LO*-2 groups. A positive integer n is said to be *square-full* if whenever the prime p divides n, p^2 divides n. The density of the set of square-full integers is also well-known [2].

Lemma 4. *Let $f(n)$ be the number of square-full integers less than or equal to n. Then*

$$f(n) \sim Cn^{1/2},$$

where C is a constant.

Let G be an *LO*-2 group of symmetric genus $g = \sigma(G)$. Then by Proposition 1, $g = 1 + 2|G|/21$, and it follows that $g = 1 + 2M/(9 \cdot 49)$, where M is a square-full integer. Thus by Lemma 4, we have the following.

Theorem 8. *Let J_2 be the set of integers g for which there is a LO-2 group of symmetric genus g. Then the set J_2 has density 0 in the set of positive integers.*

5. *LO*-3 groups.

Now we consider *LO*-3 groups. The *LO*-3 groups that are 3-groups were studied by Zomorrodian [12]. However, there are *LO*-3 groups that are not 3-groups. It is easy to find such groups in the MAGMA Small Groups Library. For example, there two *LO*-3 groups of order $3^4 \cdot 7$.

The classification of the first two terms of the lower central series of the triangle group $\Gamma(3, 3, 3n)$ was done in Lemma 2. If $\Gamma(3, 3, 3n) = \langle x, y \rangle$, then the actions of x and y on $\Gamma' = \langle u, v, w, z \rangle$ are given in equations (7) and (8). In this section we will restrict ourselves to quotients of $\Gamma(3, 3, 9)$.

Let G be an *LO*-3 group with partial presentation

$$x^3 = y^3 = (xy)^9 = 1. \tag{11}$$

Immediately, the commutator quotient group G/G' is either Z_3 or $Z_3 \times Z_3$. Both are possible, and this is the source of one of the difficulties in dealing with *LO*-3 groups.

For groups of symmetric genus one, we have the following result.

Lemma 5. *Let G be an LO-3 group. If $\sigma(G) = 1$, then $G/G' \cong Z_3 \times Z_3$.*

Proof. Since $\sigma(G) = 1$, the group G has class (c) and has a normal abelian subgroup T of index 3 [4, p. 296]. If G is a 3-group, then $G/G' \cong Z_3 \times Z_3$. Assume that G is not a 3-group. But then $|G|$ and $|T|$ are divisible by some prime $p > 3$. Since T is abelian, T has a non-trivial 3-complement Q. Then Q is characteristic in T and normal in G. Now G/Q is a 3-group of order at least 9 and has an abelian quotient of order 9. Hence $G/G' \cong Z_3 \times Z_3$.

In what follows, we will use the letters x,y,u,v,w and z both for elements of Γ and for the images of such elements in any quotient group G. The context will make clear what is meant.

Now it is not difficult to classify the LO-3 groups of genus one. We use the notation $SG(n, k)$ to denote group k of order n in the MAGMA Small Groups library.

Theorem 9. *If G is an LO-3 group with $\sigma(G) = 1$, then G is either SG(81,9) or SG(243,26).*

Proof. Let G have partial presentation (11). We know $G' = \langle u, v, w, z\rangle$. By Lemma 5, $G/G' \cong Z_3 \times Z_3$. Then G has exactly four subgroups of index 3, and these subgroups are $M_1 = \langle x\rangle G'$, $M_2 = \langle y\rangle G'$, $M_3 = \langle xy\rangle G'$ and $M_4 = \langle x^{-1}y\rangle G'$. Since $\sigma(G) = 1$, one of these four subgroups is abelian and has rank 2.

Suppose that M_1 were abelian and had rank 2. Then $u = u^x = v$, $v = v^x = u^{-1}v^{-1}$ and so $o(u) = 3$. The same argument says that $z = w$ and $o(w) = 3$. Then M_1 is generated by elements of order 3 so that G is a 3-group with $|G| \leq 27$. This is not possible, since G is an LO-3 group. Hence M_1 cannot be abelian. In the same way, M_2 is not abelian either.

Suppose that M_4 were abelian. Then $yx^{-1} = x(x^{-1}y)x^{-1} \in M_4$. Then $x^{-1}y \cdot yx^{-1} = yx^{-1} \cdot x^{-1}y$. But this gives $(yx)^3 = 1$, which is a contradiction, since $o(yx) = o(xy) = 9$. Thus M_4 cannot be abelian.

Therefore, $M_3 = \langle xy\rangle G' = \langle xy, yx\rangle$ is abelian, generated by two elements of order 9. Now the LO-3 group G is a 3-group of order $3|M_3| \leq 243$. But $|G|$ cannot be less than 81. Hence $|G|$ is 81 or 243. A survey of the groups of orders 81 and 243 using MAGMA shows that G is either $SG(81, 9)$ or $SG(243, 26)$.

Another source of difficulty in dealing with LO-3 groups is that if G is an LO-3 group with a normal subgroup N of index greater than 3, it may not be true that G/N is an LO-3 group. Let G be an LO-3 group with partial presentation (11), and let N be a proper normal subgroup of G. Obviously, if x, y or xy is in N, then $[G : N] = 3$. We record the following easy result.

Lemma 6. *Let G be an LO-3 group with partial presentation (11). Let N be a normal subgroup with $[G : N] > 3$, and let $Q = G/N$.*
(1) If $(xy)^3 \in N$, then $\sigma(Q) = 1$.
(2) If $(xy)^3 \notin N$, then Q is an LO-3 group.

Corollary 5. *If 3 does not divide $|N|$, then G/N is an LO-3 group.*

We also note the following basic result about the order of an *LO*-3 group.

Proposition 7. *If G is an LO-3 group, then 81 divides $|G|$.*

Proof. First, we know that 81 divides $|G|$ if G is a 3-group. Assume that G is not a 3-group. We know that 9 divides $|G|$ and $|G/G'|$ is 3 or 9. Also, 3 divides $|G'|$, by Corollary 5.

Assume to the contrary that 81 does not divide $|G|$, and let P be the Sylow 3-subgroup of G'. If $|G/G'| = 9$, then we must have $|P| = 3$. If $|G/G'| = 3$, then $|P|$ is either 3 or 9. In any case, P has a normal 3-complement L in G', as a consequence of Burnside's theorem, since 3 is the smallest prime dividing $|G'|$ [10, pp. 139, 141]. But L is characteristic in G' and thus normal in G. Then the 3-group G/L is an *LO*-3 group by Corollary 5, which is a contradiction, since $|G/L|$ is 27 or less. Hence 81 divides $|G|$.

Next we consider the analogs of Corollary 2 and Theorem 6 for *LO*-3 groups. First, by Corollary 1 we have the following.

Corollary 6. *Let G be a finite LO-3 group and let p be a prime congruent to 2 (mod 3). If p divides $|G|$, then p^2 divides $|G|$.*

Let G be an *LO*-3 group. Then by Corollary 6 and Proposition 7, the order of G has the form

$$|G| = 3^i \cdot p_1^{n_1} \cdots p_s^{n_s} \cdot q_1^{m_1} \cdots q_t^{m_t}, \tag{12}$$

where $i \geq 4$, each p_j is a prime congruent to 1 (*mod* 3) (possibly $s = 0$), each q_j is a prime congruent to 2 (*mod* 3), and each exponent $m_j \geq 2$ (possibly $t = 0$). Then by Proposition 1, the symmetric genus $g = \sigma(G) = 1 + |G|/9$, and thus $g - 1$ is the product of primes that are not congruent to 2 (*mod* 3) and an integer that is "square-full" of primes that are congruent to 2 (*mod* 3). Let $f(n)$ be the number of integers of this form that are less than or equal to n. We use results from analytic number theory to obtain an upper bound for $f(n)$.

Lemma 7. *The number of integers n, $n \le x$, that are not divisible by any prime $p \equiv 2$ $(mod\ 3)$ does not exceed $Cx(\log x)^{-1/2}$, for some constant C.*

Proof. This is a well-known result in analytic number theory. It can be established by combining Theorem 3.6 and Corollary 4.12(d) in [9].

Lemma 8. *Let B be the set of all square-full integers greater than a fixed positive integer y. Then $\sum_{b\in B}\frac{1}{b} \le \frac{C}{\sqrt{y}}$, for some constant C.*

Proof. We begin by converting the series to a Riemann-Stieltjes integral over the function $\alpha(t) = \sum_{y<b\le t} 1$, where the values of b are the square-full integers in this range. Note that $\alpha(t) \le C\sqrt{t}$. Next we use integration by parts and derive the following.

$$\sum_{b\in B}\frac{1}{b} = \int_y^\infty \frac{1}{t}\,d\alpha = [\frac{1}{t}\alpha(t)]_y^\infty + \int_y^\infty \frac{\alpha(t)}{t^2}dt \le \lim_{t\to\infty}\frac{C\sqrt{t}}{t} + \int_y^\infty Ct^{-3/2}dt$$

The result follows by integration.

Lemma 9.

$$f(n) \le \frac{Cn}{(log\ n)^{1/2}},$$

for some constant C.

Proof. Assume that a is an integer that is a product of primes not congruent to 2 $(mod\ 3)$ and that b is an integer that is a product of squares of primes that are congruent to 2 $(mod\ 3)$. Then every number counted in $f(n)$ is a product of the form $a \cdot b$. By Lemma 7

$$f(n) = \sum_{ab\le n} 1 = \sum_{b\le\sqrt{n}}\ \sum_{a\le\frac{n}{b}} 1 + \sum_{b>\sqrt{n}}\ \sum_{a\le\frac{n}{b}} 1 \le \sum_{b\le\sqrt{n}} \frac{C\cdot\frac{n}{b}}{\sqrt{log(\frac{n}{b})}} + \sum_{b>\sqrt{n}}\frac{n}{b}.$$

Now by Lemma 8, $\sum_{b>\sqrt{n}}\frac{n}{b} = n\cdot\sum_{b>\sqrt{n}}\frac{1}{b} \le D\cdot\frac{n}{\sqrt[4]{n}}$, for some constant D.

Also, for $b \le \sqrt{n}$, we have that $\log(\frac{n}{b}) \ge \log(\sqrt{n})$ and

$$\sum_{b\le\sqrt{n}} \frac{C\cdot\frac{n}{b}}{\sqrt{log(\frac{n}{b})}} \le \frac{Cn}{\sqrt{.5\log(n)}}\cdot\sum_{b\le\sqrt{n}}\frac{1}{b}.$$

By Lemma 8, the series $\sum_{b\le\sqrt{n}}\frac{1}{b}$ is bounded. It follows that

$$f(n) \le \frac{C_1 n}{\sqrt{\log(n)}} + D\cdot\frac{n}{\sqrt[4]{n}},$$

for some constant C_1. The result follows.

An immediate consequence of Lemma 9 is the following.

Theorem 10. *Let J_3 be the set of integers g for which there is a LO-3 group of symmetric genus g. Then the set J_3 has density zero in the set of positive integers.*

Combining Theorems 6, 7, 8 and 10 yields Theorem 1.
Also, Lemma 9 and Corollary 1 prove Theorem 2.

6. Metabelian *LO*-3 groups.

Now we consider the metabelian *LO*-3 groups because they are important in questions about the genus spectrum. We have a restriction on the orders of *LO*-3 in equation (12). Suppose that n is any integer of the form $n = 3^i \cdot \prod_{p \in R} p^{\ell_p} \cdot \prod_{q \in S} q^{m_q}$, where $i \geq 4$, R is a finite set of primes of the form $p \equiv 1 \ (mod\ 3)$ and S is a finite set of primes of the form $q \equiv 2 \ (mod\ 3)$. By (12), the exponents m_q must be at least 2. We will show that if all the exponents m_q are even, then there exists a metabelian *LO*-3 group with $|G| = n$. One obvious consequence is that if n is an integer divisible by 81 and all other primes that divide n are of the form $p \equiv 1 \ (mod\ 3)$, then n is the order of a metabelian *LO*-3 group.

Definition 1. *Let $\Delta(3,3,9) = \Gamma(3,3,9)/\Gamma(3,3,9)''$.*

Every metabelian *LO*-3 group is a quotient of the infinite group $\Delta(3,3,9)$. Next we will construct a family of finite quotient groups of $\Delta(3,3,9)$ so that every metabelian *LO*-3 group is a quotient of one of these groups.

Definition 2. *Let $G(m)$ be the group $\Delta(3,3,9)/N$ where N is the normal closure of the subgroup $\langle ([y^{-1}, x^{-1}])^m \rangle = \langle u^m \rangle$ in $\Delta(3,3,9)$.*

The group $G(m)$ is a finite metabelian quotient of the $(3,3,9)$ triangle group. Let $H = \Delta(3,3,9)$. Using equations (7) and (8), it is easy to show that $N = \langle u^m, v^m, w^m, z^m \rangle = (H')^m$. Now $G(m) = H/N$. Notice that if 3 does not divide m, the relation $(vw)^3 = 1$ implies that $v = w^{-1}$ and similarly $u = z^{-1}$. In this case, $|G| = 9m^2$ and by Proposition 7, G is not an *LO*-3 group. Therefore, we will consider the groups $G(3n)$. The group $G(3n)$ has order $729n^2$.

Theorem 11. *Let G be a finite metabelian LO-3 group. Then G is a quotient of the group $G(3n)$ for some integer n.*

Proof: The order of any of the commutators, u, v, w and z in G is the same and 3 divides $m = |u|$. So, $m = 3n$ and it follows that G satisfies all of the relations of $G(3n)$. Hence G is a quotient of that group.

Let $p > 3$ be a prime that divides the integer n. Let P be a Sylow p-subgroup of $G(3n)$. Since P is contained in $G'(3n) \cong Z_3 \times Z_3 \times Z_{3n} \times Z_{3n}$, it is an abelian p-group of rank 2 and is normal in $G(3n)$. Therefore, any finite metabelian LO-3 group G has Sylow p-subgroups that are abelian of rank 2 or less.

Next we determine the center of $G(3n)$.

Proposition 8. *Consider the group $G(3n)$ for some integer n.*
(1) if 3 does not divide n, then $Z(G(3n)) \cong Z_3$, and
(2) if 3 divides n, then $Z(G(3n)) \cong Z_3 \times Z_3$.

Proof: The group $\langle uz, vw \rangle$ is normal in G by (9) and isomorphic to $Z_3 \times Z_3$. Also $\langle uz, vw, u, v \rangle = G'$. Let $G = G(3n)$ and suppose that $a = u^\lambda v^r w^s z^t \in G'$, where $-1 \le \lambda \le 1$, $-1 \le r \le 1$, $0 \le s \le 3n$ and $0 \le t \le 3n$. Now $a \in Z(G)$ if and only if $a^x = a$ and $a^y = a$. From $a^x = a$, we deduce that $\lambda + r \equiv 0 \ (mod\ 3)$, $\lambda + r \equiv t + s \ (mod\ 3n)$, $2r - \lambda \equiv 0 \ (mod\ 3)$ and $2r - \lambda \equiv 2s - t \ (mod\ 3n)$. Therefore, $r \equiv s \ (mod\ n)$ and $\lambda \equiv t \ (mod\ n)$. The same congruences can be deduced from $a^y = a$. Now we see that $r = -\lambda$. It follows that $s + t \equiv 0 \ (mod\ 3n)$ and so $s = -t$. Also $t = \lambda + kn$ for some integer k and so $s = -t = -\lambda - kn$. Now for $-1 \le \lambda \le 1$ and $0 \le k \le 2$, we have

$$a = u^\lambda v^{-\lambda} w^{-\lambda - kn} z^{\lambda + kn} = (uv^{-1}w^{-1}z)^\lambda (w^{-1}z)^{kn}.$$

It is easy to check that $\langle uv^{-1}w^{-1}z \rangle \subseteq Z(G)$. Next, $(w^{-n}z^n)^x = w^{-n}z^n$ and $(w^{-n}z^n)^y = u^{-2n}v^{-n}w^n z^{-n}$.

Now, if 3 divides n, then since $u^3 = z^{-3}$, $u^{2n} = z^{-2n}$ and similarly, $v^{-n} = w^n$. Therefore, if 3 divides n, then $(w^{-n}z^n)^y = w^{2n}z^n = w^{-n}z^n$ and $w^{-n}z^n \in Z(G)$.

Suppose that 3 does not divide n. If $w^{-n}z^n \in Z(G)$, then $(u^n z^n)^2 = (v^n w^n)^{-1}$. These two elements are independent and so both must be trivial. Therefore, $(uz)^{2n} = (uz)^3 = 1$ and $u = z^{-1}$, which can't happen. This completes the proof.

Theorem 12. *Let G be a finite metabelian LO-3 group. If $|G| = p^k m$ for some prime $p > 3$, where $gcd(p, m) = 1$ and k is odd, then* 3 *divides* $(p - 1)$.

Proof: Suppose that G is the image of a $(3, 3, 9)$ triangle group with generators x and y. Let P be the Sylow p-subgroup of G. Therefore,

$P \subseteq G'$ and P has rank 2 or less. Let $u = [y^{-1}, x^{-1}] \in G$ and suppose that $n = o(u)$. Therefore, G is a quotient of $G(n)$ by a normal subgroup N. Let Q be the Sylow p-subgroup of $G(n)$. Now $|Q| = p^{2\ell}$ and so $Q \cap N$ is a normal subgroup of $G(n)$ satisfying $|Q \cap N| = p^t$, where t is odd. If $Q \cap N$ has rank 1, then $(Q \cap N)^{p^{t-1}}$ is a normal subgroup of order p. If $Q \cap N$ has rank 2, then $Q \cap N \cong Z_{p^r} \times Z_{p^s}$, where $r < s$. In this case, $(Q \cap N)^{p^{s-1}}$ is a normal subgroup of order p. Since the normal subgroup of order p is not in the center of $G(n)$, $G(n)$ must act on it non-trivially. It follows that 3 divides $(p-1)$.

Corollary 7. *Let G be a finite metabelian LO-3 group, and let q be a prime congruent to 2 $(mod\ 3)$. If $|G| = q^k m$, where $gcd(q, m) = 1$, then k is even.*

Primes congruent to 1 $(mod\ 3)$ can occur to odd powers in the order of a metabelian LO-3 group. For example, $SG(567, 17)$ is an LO-3 group of order $3^4 \cdot 7$.

Lemma 10. *let p be a prime congruent to 1 $(mod\ 3)$. Let $G = G(n)$, with $n = p^k m$, where $gcd(p, m) = 1$ and 3 divides m. There exist two normal subgroups N of order p so that G/N is an LO-3 group.*

Proof: Let u and v be the commutators defined in G. Define $\overline{u} = u^{mp^{k-1}}$ and the other powers of the commutators similarly. Therefore $\overline{u}$ and the rest have order p.

Since $Aut(Z_p) \cong Z_{p-1}$, there is a unique subgroup $\langle \alpha \rangle$ of $Aut(Z_p)$ of order 3. If $Z_p = \langle c \rangle$, then $\alpha(c) = c^t$, where $t^3 \equiv 1$ $(mod\ p)$ and t is not congruent to 1 $(mod\ p)$. There are two possible values of t. Notice that $t + 1 \equiv -t^2$ $(mod\ p)$.

Define $N = \langle \overline{u}(\overline{v})^{-t} \rangle$. Also since 3 divides m, $\overline{z} = (\overline{u})^{-1}$ and $\overline{w} = (\overline{v})^{-1}$. Now

$$(\overline{uv}^{-t})^x = \overline{v}(\overline{u}^{-1}\overline{v}^{-1})^{-t} = (\overline{u}^t\overline{v}^{t+1}) = (\overline{u}^t\overline{v}^{-t^2}) = (\overline{uv}^{-t})^t,$$

and

$$\begin{aligned}(\overline{uv}^{(-t)})^y = \overline{w}^{-1}(\overline{wz})^{-t} = (\overline{w}^{(-1-t)}\overline{z}^{(-t)}) = (\overline{w}^{t^2}\overline{z}^{(-t)}) = (\overline{u}^t\overline{v}^{(-t^2)}) \\ = (\overline{uv}^{(-t)})^t.\end{aligned}$$

It follows that N is a normal subgroup of $G(n)$ of order p and there is one such subgroup for each choice of t.

Lemma 11. *Let $G = G(n)$, where $n = 3m$, There exists a normal subgroup N of order 9 so that G/N is an LO-3 group.*

Proof: Let $N = \langle uv^{-1}w^{-1}z, w^m z^{-m}\rangle$. Note that $uv^{-1}w^{-1}z \in Z(G)$. Now $(w^m z^{-m})^x = (z^{-m}w^{-m})w^{-m} = w^m z^{-m}$, since $o(w^m) = 3$. Also

$$(w^m z^{-m})^y = (u^m w^{-m})(u^m v^m z^m) = (uv^{-1}w^{-1}z)^{-m}(w^m z^{-m}).$$

It follows that N is normal in G and $|N| = 9$. It is not difficult to see that $vw \notin N$. Since $(vw)^x = (xy)^3$, we see that $o(xN) = 3$, $o(yN) = 3$ and $o(xyN) = 9$. This shows that G/N is an LO-3 group.

Now we classify the integers that are orders of metabelian LO-3 groups, and, consequently, the integers that are the genus values of metabelian LO-3 groups.

Theorem 13. *Let R be a finite set of primes congruent to* 1 (*mod* 3) *and S a finite set of primes congruent to* 2 (*mod* 3)*. For each prime $p \in R$, let ℓ_p be a positive integer, and for each prime $q \in S$, let $2k_q$ be a positive even integer. Finally choose an integer $i \geq 4$. Then there exists a metabelian LO-3 group G of order*

$$3^i \cdot \prod_{p\in R} p^{\ell_p} \cdot \prod_{q\in S} q^{2k_q}. \tag{13}$$

Furthermore, the order of every metabelian LO-3 group has this form.

Proof: For each number ℓ_p, define $\alpha_p = \ell_p/2$ if ℓ_p is even, and $\alpha_p = (\ell_p + 1)/2$ if ℓ_p is odd. Next define $\beta = 1$ if $i \leq 6$. If $i \geq 6$, then let $\beta = (i - 4)/2$ if i is even and $\beta = (i - 3)/2$ if i is odd. Next define $n = 3^\beta \cdot \prod_{p\in R} p^{\alpha_p} \cdot \prod_{q\in S} q^{k_q}$. Note that 3 divides n. The group $G(n)$ has order $81n^2 = 3^{4+2\beta} \cdot \prod_{p\in R} p^{2\alpha_p} \cdot \prod_{q\in S} q^{2k_q}$. For every prime $p \in R$ such that ℓ_p is odd, we can find a normal subgroup N_p of order p by Lemma 10. Depending on the size of i, we can choose the normal subgroup N_3 of order 1, 3 or 9, by using either $Z(G(n))$ or Lemma 11. The product of all of these normal subgroups is a normal subgroup N and the quotient $G(n)/N$ is an LO-3 group of order (13).

That the order of a metabelian LO-3 group has the form (13) is a consequence of Proposition 7 and Corollary 7.

It is important to note the difference between the possible orders for LO-3 groups in equation (12) and the orders for metabelian LO-3 groups in Theorem 13. The obvious question is whether every possible order for an LO-3 group is realized by a metabelian LO-3 group. In [7], every positive integer was shown to be the strong symmetric genus of a group of a very restricted type. Also, Conder and Tucker conjecture in [1, p. 285] that every number is the symmetric genus of a finite abelian or metabelian group. With regard to *LO*-3 groups, we make the following conjecture (though we admit that we have very little supporting evidence).

Conjecture 1. *Let g be the genus of an LO-3 group. Then there exists a metabelian LO-3 group G satisfying $g = \sigma(G)$.*

The smallest *LO*-3 group that is not metabelian is $G = SG(1053, 51)$. This group has a very interesting structure, but, as far as we can tell, it is not part of an infinite family of groups. Also G/G'' is not an *LO*-3 group.

We have also constructed examples of *LO*-3 groups with derived length 3. The smallest of these groups has partial presentation (11), with the definition of u, v, w and z as commutators as indicated and the added relations $u^{15} = 1$, $[u, v] = 1$ and $[[u, z], u] = [[u, z], v] = [[u, z], w] = [[u, z], z] = 1$. This group G has order $492075 = 3^9 \cdot 5^2$ and $|G''| = 27$. Changing $u^{15} = 1$ to $u^{33} = 1$ results in a group of order $2381643 = 3^9 \cdot 11^2$ and $|G''| = 27$. These groups are extensions of metabelian *LO*-3 groups. Unfortunately, none of these examples shed any light on Conjecture 1.

Let's finish this discussion with some comments about metabelian quotients of $\Gamma(3, 3, 3n)$.

Definition 3. *Let $\Delta(3, 3, 3n) = \Gamma(3, 3, 3n)/\Gamma(3, 3, 3n)''$. Let $G_n(m)$ be the group $\Delta(3, 3, 3n)/N$ where N is the normal closure of the subgroup $\langle([y^{-1}, x^{-1}])^m\rangle = \langle u^m\rangle$ in $\Delta(3, 3, 3n)$.*

This definition mirrors Definition 2. By Lemma 2, if n divides m, then $|G_n(m)| = 9n^2m^2$. The calculation of the center $Z(G_n(m))$ needs to be redone. Some Magma calculations suggest the following conjecture.

Conjecture 2. *Consider the group $G_n(m)$ for n dividing m.*
(1) if 3 does not divide m, then $Z(G_n(m))$ is trivial,
(2) if 3 divides m and 3 does not divide n, then $Z(G_n(m)) \cong Z_3$, and
(3) if 3 divides n, then $Z(G(3n)) \cong Z_3 \times Z_3$.

Likewise, the normal subgroup calculations need to be redone. However, we expect an analog of Theorem 13 to be true.

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