THE STRONG SYMMETRIC GENUS SPECTRUM OF ABELIAN GROUPS

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Abstract. Let $\mathcal{S}$ denote the set of positive integers that may appear as the strong symmetric genus of a finite abelian group. We obtain a set of (simple) necessary and sufficient conditions for an integer g to belong to $\mathcal{S}$. We also prove that the set $\mathcal{S}$ has an asymptotic density and estimate its value.

1. Introduction

Let G be a finite group. Among the various genus parameters associated with G, the most classical is perhaps the *strong symmetric* genus $\sigma^0(G)$, the minimum genus of any Riemann surface on which G acts faithfully and preserving orientation. Work on this parameter dates back over a century and includes the fundamental bound $\sigma^0(G) \le 84(g-1)$ due to Hurwitz [4].

A natural problem is to determine the positive integers that occur as the strong symmetric genus of a group (or a particular type of group), that is, to determine the strong symmetric genus spectrum for the particular type of group. This basic problem was settled for the family of all finite groups by May and Zimmerman [6]: there is a group of strong symmetric genus g, for all $g \in \mathbb{N}$. Our focus here is to describe the strong symmetric genus spectrum of abelian groups.

Let

$$\mathcal{S} = \left\{ g \in \mathbb{N} : g = \sigma^0(A) \text{ for some abelian group } A \right\}$$

denote the strong symmetric genus spectrum of abelian groups. Henceforth, we will refer to $\mathcal{S}$ simply as the "spectrum." The abelian groups of strong symmetric genus zero are exactly the cyclic groups, and those of strong symmetric genus one are exactly the abelian groups of rank 2 and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. These facts are a direct consequence of the classification of the groups of strong symmetric genus zero or one (see Gross and Tucker [2, §6.3]). One can find the strong symmetric genus of any other abelian group by applying a classical result due to Maclachlan [5, Theorem 4]. Recall that every finite abelian group A has a canonical representation $A \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$, with standard invariants $m_1, m_2, \dots, m_r$ subject to $m_1 > 1$ and $m_i | m_{i+1}$ for $1 \le i < r$. If we extend the list of standard invariants by adding $m_0 = 1$ to it, we can state Machlachlan's theorem as follows.

Theorem M (Maclachlan, 1965). *Let A be a finite abelian group, with $|A| \ge 10$, and let $m_0 = 1, m_1, \dots, m_r$, $r \ge 3$, denote the standard invariants of A. Then*

$$\sigma^0(A) = 1 + \frac{|A|}{2} \min_{0 \le \gamma \le r/2} \left\{ 2\gamma - 2 + \sum_{i=1}^{r-2\gamma} \left(1 - \frac{1}{m_i}\right) + \left(1 - \frac{1}{m_{r-2\gamma}}\right) \right\}. \tag{1}$$

When $a > 1$ and $a^3 n \ge 10$, Maclachlan's formula yields

$$\sigma^0(\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}) = 1 - a^2 + a^2(a-1)n. \tag{2}$$

In particular, when $a = 2$, this reveals that $\mathcal{S}$ contains the entire residue class $g \equiv 1 \pmod 4$. It is also not difficult to deduce from (1) that $\sigma^0(A) - 1$ cannot be a squarefree integer.

Theorem 1. *If $g \ge 2$ and $g - 1$ is squarefree, then $g \notin \mathcal{S}$.*

Date: August 24, 2015.

Our first objective is to take such observations a step further and to provide a relatively simple test that can be used to check whether a given positive integer g belongs to the spectrum $\mathcal{S}$. In §2, we establish the following result.

Theorem 2. *Let $g \geq 2$. Then $g \in \mathcal{S}$ if and only if g satisfies one of the following conditions:*

(i) $g \equiv 1 \pmod 4$ *or* $g \equiv 55 \pmod{81}$*;*
(ii) $g - 1$ *is divisible by* p^4 *for some odd prime p;*
(iii) $g - 1$ *is divisible by* a^2 *for some odd integer a such that* $(a - 1) \mid g$*;*
(iv) $g - 1$ *is divisible by* $b^2a^2(a - 1)$ *for some odd integers* $a, b > 1$*, with* $a \equiv 3 \pmod 4$*.*

We remark that each of conditions (ii)–(iv) can be checked easily given the prime factorization of the integer $g - 1$. Therefore, the above theorem does provide a reasonable test to check whether a specific integer g is in the spectrum. As part of the proof of Theorem 2 we also establish the following result, which of independent interest.

Theorem 3. *Suppose that A is an abelian group of rank 5 or higher. Then there exists an abelian group B of rank 3 or 4 such that $\sigma^0(A) = \sigma^0(B)$.*

If $\mathcal{A}$ is a set of integers, its *lower* and *upper asymptotic densities*, denoted $\underline{\delta}(\mathcal{A})$ and $\overline{\delta}(A)$, are given by

$$\underline{\delta}(\mathcal{A}) = \liminf_{X\to\infty} X^{-1}A(X) \qquad \text{and} \qquad \overline{\delta}(\mathcal{A}) = \limsup_{X\to\infty} X^{-1}A(X),$$

where $A(X) = |\mathcal{A} \cap [1, X]|$. A set $\mathcal{A}$ is said to have an *asymptotic density*, if $\underline{\delta}(\mathcal{A}) = \overline{\delta}(A)$; when $\mathcal{A}$ does have an asymptotic density, it is denoted $\delta(\mathcal{A})$. Since the set of squarefree integers is known to have an asymptotic density of $6\pi^{-2} \approx 0.6079$ (see Montgomery and Vaughan [8, Theorem 2.2]), we find as a direct corollary of Theorem 1 that $\overline{\delta}(\mathcal{S}) \leq 0.3921$. On the other hand, since all the integers $g \equiv 1 \pmod 4$ are in the spectrum, we have $\underline{\delta}(\mathcal{S}) \geq 0.25$. It is therefore natural to ask whether the spectrum $\mathcal{S}$ has an asymptotic density—which is not obvious—and what its potential value is. The second main result of the paper establishes that the asymptotic density does indeed exist.

Theorem 4. *The spectrum $\mathcal{S}$ has an asymptotic density $\delta(\mathcal{S}) \approx 0.3284$.*

We remark that this shows that the lower bound $\underline{\delta}(\mathcal{S}) \geq 0.3175...$ given by Borror, Morris and Tarr [1] is quite tight.

2. The structure of $\mathcal{S}$

Recall that abelian groups of ranks one or two have strong symmetric genus zero or one, respectively. Henceforth, we focus on groups of ranks three and higher. Throughout this section, we assume the notation of Theorem M. In particular, for an abelian group A of rank $r \geq 3$, we write $m_1, m_2, \dots, m_r$ for its standard invariants and assume that $m_i \mid m_{i+1}$.

2.1. **Proof of Theorem 1.** Let A be an abelian group of rank $r \geq 3$, with $\sigma^0(A) = g \geq 2$. The main idea behind Theorem 1 is the observation that if a prime p divides m_1, the smallest invariant of A, then (1) forces p^2 to divide $g - 1$. However, while this observation can be fully justified, we opt for brevity and deduce the theorem from the following general result on actions of p-groups by May and Zimmerman [7, Theorem 3].

Lemma 1. *Let p be a prime number, and let G be a non-cyclic p-group that acts on a Riemann surface of genus $g \geq 2$ preserving orientation. Suppose that a largest cyclic subgroup of G has index p^t.*

(i) *If p is odd, then* $g \equiv 1 \pmod{p^t}$*.*
(ii) *If* $p = 2$ *and* $t > 1$*, then* $g \equiv 1 \pmod{2^{t-1}}$*.*

Let X be a Riemann surface of genus g such that A acts on X preserving orientation. Then any subgroup of A also acts on X. In particular, this is true for its Sylow p-subgroup A_p, where p is any odd prime that divides m_1. Since the largest cyclic subgroup of A_p has index p^t, $t \geq 2$, it immediately follows from Lemma 1(i) that $g \equiv 1 \pmod{p^2}$.

When $m_1 = 2^k$, $k \geq 1$, we may apply the above argument to the Sylow 2-subgroup of A. If 2^t, $t \geq 2$, is the index of its largest cyclic subgroup, Lemma 1(ii) yields $g \equiv 1 \pmod 4$, provided that $t \geq 3$. That leaves the case when $t = 2$. In this case, we must have $r = 3$, $m_1 = 2$, $m_2 = 2a$, and $m_3 = 2an$, with a odd. Thus, Maclachlan's formula (1) gives

$$g = 1 + 4a \min\left\{\tfrac{1}{2}(3a-1)n - 1, an\right\} \equiv 1 \pmod 4,$$

since $\frac{1}{2}(3a-1)$ is an integer. This completes the proof of Theorem 1.

2.2. **Proof of Theorem 2.** To begin, recall that all integers $g \equiv 1 \pmod 4$ are part of the spectrum $\mathcal{S}$, by (2) with $a = 2$. Similarly to (2), Maclachlan's theorem gives also

$$\sigma^0(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}) = 81n - 26, \tag{3}$$

so the congruence class $g \equiv 55 \pmod{81}$ is also part of $\mathcal{S}$.

Further, by (2), the spectrum $\mathcal{S}$ contains any natural number g satisfying a congruence of the form

$$g \equiv 1 - a^2 \pmod{a^2(a-1)}$$

for some odd $a > 1$. These are exactly the integers described by condition (iii) of Theorem 2, since we may apply the Chinese Remainder Theorem to rewrite the above congruence as the pair of congruences

$$g \equiv 1 \pmod{a^2}, \qquad g \equiv 0 \pmod{a-1}.$$

2.2.1. *The spectrum of groups of rank 3.* In this section, we study genera of abelian groups of rank 3 and establish the following result.

Proposition 1. *The spectrum of abelian groups of rank* 3 *consists of the congruence class* $g \equiv 1 \pmod 4$ *and the integers* g *satisfying conditions (iii) or (iv) of Theorem 2.*

As we noted already, the spectrum of the groups of type $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2n}$ is the residue class $g \equiv 1 \pmod 4$, and the spectrum of the groups of type $\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}$, with a odd, are the integers g satisfying condition (iii) of Theorem 2. We also observe, for future reference, that when $a > 1$ is odd, (1) gives

$$\sigma^0(\mathbb{Z}_a \times \mathbb{Z}_{2a} \times \mathbb{Z}_{2a}) = \sigma^0(\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{4a}) = 1 - 5a^2 + 4a^3. \tag{4}$$

Thus, the spectrum of groups of type $\mathbb{Z}_a \times \mathbb{Z}_{2a} \times \mathbb{Z}_{2a}$ is contained in the integers satisfying condition (iii) of Theorem 2.

Next, we consider a general abelian group of rank 3 and write its canonical form as $\mathbb{Z}_a \times \mathbb{Z}_{ab} \times \mathbb{Z}_{abn}$, where a, b, n are positive integers with $a > 1$. Our first order of business is to show that the total contribution to the spectrum of such groups with even a is the congruence class $g \equiv 1 \pmod 4$. The next lemma establishes that and a little more.

Lemma 2. *Let A be a finite abelian group. If the Sylow* 2*-subgroup of A has rank* 3 *or higher, then* $\sigma^0(A) \equiv 1 \pmod 4$.

Proof. The proof is similar to the proof of Theorem 1. Let A act on a Riemann surface X of genus $g = \sigma^0(A) \geq 2$ preserving orientation, and assume that its Sylow 2-subgroup A_2 has rank at least 3. Let 2^t, $t \geq 2$, be the index of a largest cyclic subgroup of A_2. Since A_2 acts on X as well, we can apply Lemma 1(ii) to get $g \equiv 1 \pmod{2^{t-1}}$. When $t \geq 3$, this establishes the lemma.

When $t = 2$, we must have $\mathrm{rank}(A_2) = 3$. Further, the invariants $m_1, \ldots, m_{r-3}$ are all odd and m_{r-2}, m_{r-1}, m_r are even, but m_{r-1} and m_{r-2} are not divisible by 4. By Theorem M, there exists an integer $\gamma \geq 0$ such that

$$\sigma^0(A) = 1 + \frac{1}{2}\left\{2(\gamma - 1)n + \sum_{i=1}^{r-2\gamma}(n - n_i) + (n - n_{r-2\gamma})\right\}, \tag{5}$$

where $n = |A| = m_1 \cdots m_r$ and $n_i = n/m_i$. We note that $n, n_1, \ldots, n_{r-3}$ are divisible by 8. Hence, the expression in the braces on the right side of (5) reduces modulo 8 to one of the following:

$$0, \quad -2n_{r-2}, \quad -n_{r-2} - n_{r-1} - 2n_r.$$

Since all three of n_{r-2}, n_{r-1} and n_r are divisible by 4 and n_{r-1} and n_{r-2} are divisible by the same power of 2, we have

$$2n_{r-2} \equiv n_{r-2} + n_{r-1} + 2n_r \equiv 0 \pmod 8,$$

and the lemma follows from (5). □

In view of Lemma 2 and (4), we may now focus on groups of type $\mathbb{Z}_a \times \mathbb{Z}_{ab} \times \mathbb{Z}_{abn}$, with an odd $a > 1$, $b > 1$, and $bn > 2$. The next lemma describes the spectrum of such groups.

Lemma 3. *Let $A \cong \mathbb{Z}_a \times \mathbb{Z}_{ab} \times \mathbb{Z}_{abn}$, $a > 1$, be an abelian group of rank 3 in canonical form. If a is odd, $b > 1$, and $bn > 2$, then either $\sigma^0(A) \equiv 1 \pmod 4$, or $\sigma^0(A) = 1 + b^2a^2(a-1)n$, with b odd and $a \equiv 3 \pmod 4$. In particular, the spectrum of such groups contains the integers g satisfying condition (iv) of Theorem 2.*

Proof. By Theorem M,

$$\sigma^0(A) = 1 + a^3b^2n \min\left\{1 - \frac{1}{2a} - \frac{1}{2ab} - \frac{1}{abn}, 1 - \frac{1}{a}\right\}. \tag{6}$$

Under the hypotheses $b > 1$ and $bn > 2$, we have

$$\frac{1}{2ab} + \frac{1}{abn} \leq \min\left\{\frac{1}{4a} + \frac{1}{4a}, \frac{1}{6a} + \frac{1}{3a}\right\} = \frac{1}{2a}.$$

Therefore, (6) yields $\sigma^0(A) = 1 + b^2a^2(a-1)n$. Finally, when b is even or $a \equiv 1 \pmod 4$, we have $\sigma^0(A) \equiv 1 \pmod 4$. □

We can now summarize our findings about the spectrum of abelian groups of rank 3 as follows:

- The spectrum of groups with an even a is the congruence class $g \equiv 1 \pmod 4$.
- The spectrum of groups with an odd a and $b = 1$ consists of the even integers g satisfying condition (iii) of Theorem 2.
- The spectrum of groups with an odd a and $b > 1$, where either $a \equiv 1 \pmod 4$ or b is even, is contained in the congruence class $g \equiv 1 \pmod 4$.
- The spectrum of groups with $a \equiv 3 \pmod 4$ and an odd $b > 1$ consists of the odd integers g satisfying condition (iv) of Theorem 2.

Altogether, these observations establish Proposition 1.

We conclude this section with a useful consequence of the above discussion.

Corollary 1. *The spectrum of abelian groups of rank 3 contains the residue class $g \equiv 1 \pmod{81}$.*

Proof. By Lemma 3, the integers $g \equiv 1 \pmod{162}$ are the spectrum of the groups $\mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_{9n}$, while the integers $g \equiv 82 \pmod{162}$ are part of the spectrum $g \equiv 10 \pmod{18}$ of the groups $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}$. □

2.2.2. *The spectrum of groups of rank 4.* Now we consider abelian groups A of rank 4, writing their canonical form as $\mathbb{Z}_a \times \mathbb{Z}_{ab} \times \mathbb{Z}_{abc} \times \mathbb{Z}_{abcn}$, with $a > 1$. Since we are interested in groups with $\sigma^0(A) \not\equiv 1 \pmod 4$, we may assume, by virtue of Lemma 2, that ab is odd. The main result of this section is the following proposition.

Proposition 2. *The spectrum of abelian groups of rank* 4 *is a subset of the integers satisfying conditions (i) or (ii) of Theorem 2. Moreover, the spectrum of abelian groups of ranks* 3 *or* 4 *contains all the integers satisfying condition (ii) of Theorem 2.*

Proof. Let $A \cong \mathbb{Z}_a \times \mathbb{Z}_{ab} \times \mathbb{Z}_{abc} \times \mathbb{Z}_{abcn}$, with $a > 1$ and ab odd. By Theorem M,

$$\sigma^0(A) = 1 + |A| \min\left\{1, \frac{3}{2} - \frac{1}{2a} - \frac{1}{ab}, \frac{3}{2} - \frac{1}{2a} - \frac{1}{2ab} - \frac{1}{2abc} - \frac{1}{abcn}\right\}.$$

The assumption $a \geq 3$ leads to

$$\frac{3}{2} - \frac{1}{2a} - \frac{1}{ab} \geq \frac{3}{2} - \frac{1}{6} - \frac{1}{3} = 1.$$

Similarly, if $a \geq 5$, or $bc \geq 3$, or $c = 2$ and $n \geq 2$, we have

$$\frac{3}{2} - \frac{1}{2a} - \frac{1}{2ab} - \frac{1}{2abc} - \frac{1}{abcn} \geq 1.$$

Thus, we have $\sigma^0(A) = 1 + |A|$ for all abelian groups of rank 4 with ab odd, with the exception of the groups isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_6$. By (3), the spectrum of the former family is the congruence class $g \equiv 55 \pmod{81}$. This congruence class includes also the genus of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_6$, since

$$\sigma^0(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_6) = 298 \equiv 55 \pmod{81}.$$

Consequently, when $\sigma^0(A)$ does not satisfy condition (i) of Theorem 2, we must have $\sigma^0(A) = 1+|A|$. Since $|A|$ is divisible by p^4 for any (necessarily odd) prime divisor p of a, this establishes the first part of the proposition.

When $g = 1 + p^4 n$ for a prime $p \geq 5$, we have

$$g = \sigma^0(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{pn}),$$

and $g \in \mathcal{S}$. Together with Corollary 1, this proves the second part of the result. □

2.2.3. *Groups of higher ranks.* In the last two sections, we demonstrated that the combined spectrum of abelian groups of ranks 3 and 4 consists exactly of the integers g satisfying one of the four conditions of Theorem 2. To complete the proof of Theorem 2, we now establish Theorem 3 which states that groups of ranks 5 and higher contribute nothing more to $\mathcal{S}$.

Proof of Theorem 3. When $\sigma^0(A) \equiv 1 \pmod 4$, we may choose B of the form $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2n}$. When $\sigma^0(A) \not\equiv 1 \pmod 4$, it follows from Lemma 2 that m_1, the smallest invariant of A, must be odd. Let p be an odd prime dividing m_1, and let A_p be the Sylow p-subgroup of A. Then p divides each invariant m_i, and a largest cyclic subgroup of A_p has index p^t, with $t \geq 4$. As in the proof of Theorem 1, we may apply Lemma 1 to A_p to conclude that $\sigma^0(A) \equiv 1 \pmod{p^4}$. Thus, by Proposition 2, there exists an abelian group B of rank 3 or 4 such that $\sigma^0(B) = \sigma^0(A)$. □

3. The asymptotic density of $\mathcal{S}$

In this section, we establish Theorem 4. Our proof uses the characterization of $\mathcal{S}$ given in Theorem 2. Let $\mathcal{S}_j$, $1 \leq j \leq 4$, denote the set of integers that satisfy the jth condition of Theorem 2 but none of the previous conditions (if any). We deal with each of these four sets separately and show that each $\mathcal{S}_j$ has an asymptotic density.

Densities of residue classes play a major role in our proofs, so we begin by recalling that the residue class $x \equiv a \pmod q$ has asymptotic density $1/q$. Also, by the Chinese Remainder Theorem, the density of the intersection of two residue classes $x \equiv a_i \pmod{q_i}$, $i = 1, 2$, has density

$$\begin{cases} [q_1, q_2]^{-1} & \text{if } (q_1, q_2) \mid (a_1 - a_2), \\ 0 & \text{otherwise.} \end{cases}$$

Here and in the sequel, for integers $a, b, \dots$, we use $(a, b, \dots)$ and $[a, b, \dots]$ as abbreviations for $\operatorname{lcm}[a, b, \dots]$ and $\gcd(a, b, \dots)$, respectively. In particular, using the inclusion-exclusion principle, we see that the density of $\mathcal{S}_1$, the set of integers g satisfying condition (i) of Theorem 2, is

$$\delta_1 = \frac{1}{4} + \frac{1}{81} - \frac{1}{324} = \frac{7}{27}. \tag{7}$$

3.1. **The density of $\mathcal{S}_2$.** We split $\mathcal{S}_2$ into subsets $\mathcal{S}_{2,j}$, $2 \le j \le 4$, subject to $g \equiv j \pmod 4$. We will prove that each of these sets has asymptotic density

$$\delta(\mathcal{S}_{2,j}) = \frac{1}{4}\left(\frac{80}{81} - \frac{79}{75\zeta(4)}\right), \tag{8}$$

where $\zeta(s)$ denotes the Riemann zeta-function. Thus,

$$\delta_2 = \delta(\mathcal{S}_2) = \sum_{j=2}^{4} \delta(\mathcal{S}_{2,j}) = \frac{20}{27} - \frac{79}{100\zeta(4)} \approx .0108. \tag{9}$$

The calculation of the density (8) uses some basic facts about the distribution of biquadrate-free integers. When $k \ge 2$, let $\alpha_k(n)$ denote the characteristic function of the integers n that are not divisible by p^k for any prime p. It is well-known (and not difficult to prove) that

$$\alpha_k(n) = \sum_{d^k \mid n} \mu(d), \tag{10}$$

where $\mu(d)$ is the Möbius function and the summation is over all kth powers that divide n. One needs little more than (10) to establish the next lemma. The reader can find the details in a short paper by Prachar [9], where he establishes this result with a sharper error term.

Lemma 4. *Let $(a, q) = 1$. Then for any fixed $\varepsilon > 0$, one has*

$$\sum_{\substack{n \le X \\ n \equiv a \ (\mathrm{mod}\ q)}} \alpha_k(n) = \frac{X}{q\zeta(k)} \prod_{p \mid q} \left(1 - p^{-k}\right)^{-1} + O\left(X^{1/k+\varepsilon}\right),$$

the implied constant in the O-term depending on q and ε.

Let $T_j(X)$ denote the number of integers $g \equiv j \pmod 4$, with $g \le X$, that satisfy condition (ii) of Theorem 2. When $j = 2$ or 4, we have

$$T_j(X) = \frac{X}{4} - \sum_{\substack{h \le X \\ h \equiv j-1 \ (\mathrm{mod}\ 4)}} \alpha_4(h) + O(1),$$

and Lemma 4 yields

$$T_j(X) = \frac{X}{4}\left(1 - \frac{16}{15\zeta(4)}\right) + O\left(X^{1/3}\right). \tag{11}$$

When $j = 3$, we write $g = 2h + 1$ to get

$$T_3(X) = \frac{X}{4} - \sum_{\substack{h \le X/2 \\ h \equiv 1 \pmod 2}} \alpha_4(h) + O(1),$$

and Lemma 4 again leads to (11).

Next, let $T_j'(X)$ denote the number of integers g counted by $T_j(X)$ that satisfy also the congruence $g \equiv 55 \pmod{81}$. A variant of the above argument yields

$$T_j'(X) = \frac{X}{324}\left(1 - \frac{27}{25\zeta(4)}\right) + O\left(X^{1/3}\right). \tag{12}$$

The desired result (8) follows from (11) and (12), upon noting that the counting function $S_{2,j}(X)$ of $\mathcal{S}_{2,j}$ can be represented as

$$S_{2,j}(X) = T_j(X) - T_j'(X).$$

3.2. **The density of $\mathcal{S}_3$.** Recall that condition (iii) is equivalent to the requirement that g satisfies the congruence

$$g \equiv 1 - a^2 \pmod{a^2(a-1)} \tag{13}$$

for some odd $a > 1$. Let $\mathcal{A}$ be the set of such g, and write $\mathcal{A}(X) = \mathcal{A} \cap [1, X]$.

Let $S_3'(X)$ denote the counting function of the integers g that satisfy (13) but fail condition (ii) of Theorem 2. Note together these requirements restrict a to squarefree values. By (10),

$$\begin{aligned} S_3'(X) &= \sum_{g \in \mathcal{A}(X)} \alpha_4(g-1) + O(1) = \sum_{g \in \mathcal{A}(X)} \sum_{d^4 | (g-1)} \mu(d) + O(1) \\ &= \sum_{d \le X^{1/4}} \mu(d) \sum_{\substack{g \in \mathcal{A}(X) \\ d^4 | (g-1)}} 1 + O(1). \end{aligned}$$

Let D be a large integer. On noting that

$$\sum_{d > D} \sum_{\substack{g \in \mathcal{A}(X) \\ d^4 | (g-1)}} 1 \le \sum_{d > D} \frac{X}{d^4} \le XD^{-3},$$

we deduce that

$$S_3'(X) = \sum_{d=1}^{D} \mu(d) \sum_{\substack{g \in \mathcal{A}(X) \\ d^4 | (g-1)}} 1 + O\big(XD^{-3}\big). \tag{14}$$

To estimate the sum on the right side of (14), we first observe that the contribution from residue classes (13) with $a > A$ is bounded above by

$$\sum_{d \le D} \sum_{a > A} \frac{X}{a^2(a-1)} = O\big(XDA^{-2}\big).$$

Upon choosing $A = D^2$, we deduce that

$$S_3'(X) = \sum_{d=1}^{D} \mu(d) \sum_{a=3}^{A} \sum_{\substack{g \in \mathcal{A}_a(X) \\ d^4 | (g-1)}} 1 + O\big(XD^{-3}\big), \tag{15}$$

where $\mathcal{A}_a(X)$ is the subset of $\mathcal{A}(X)$ containing the integers g that satisfy (13) but

$$g \not\equiv 1 - b^2 \pmod{b^2(b-1)}$$

for any odd b with $1 < b < a$. We now call a set of squarefree integers $\{a_1, a_2, \dots, a_k\}$ *d-admissible* if $a_i > 1$ for all i and

$$(a_i, a_j - 1) = (a_i - 1, d) = 1 \qquad \text{for all } i, j \in \{1, 2, \dots, k\}. \tag{16}$$

The d-admissibility of a set $\{a_1, a_2, \dots, a_k\}$ means that the congruences (13) with $a = a_i$, $1 \le i \le k$, are consistent with one another and also with the condition $d^4 \mid (g-1)$. In particular, if $\{a\}$ is d-admissible (i.e., if $(a-1, d) = 1$), the set $\mathcal{A}_a$ has density $\delta(d, a)$, given by

$$\delta(d, a) = \frac{1}{[d^4, f(a)]} - \sideset{}{'}\sum_{b<a} \frac{1}{[d^4, f(a), f(b)]} + \sideset{}{'}\sum_{b_2<b_1<a} \frac{1}{[d^4, f(a), f(b_1), f(b_2)]} - \cdots,$$

where $f(x) = x^2(x-1)$ and the summations are over odd integers $b, b_1, b_2, \dots$ such that the sets $\{a, b\}, \{a, b_1, b_2\}, \dots$ are d-admissible. The same application of the inclusion-exclusion principle that leads to the last formula also lets us rewrite (15) as

$$S_3'(X) = X \sum_{d=1}^{D} \mu(d) \sideset{}{'}\sum_{a=3}^{A} \delta(d, a) + O\big(D 2^A + X D^{-3}\big).$$

Hence, if we choose $D = \lfloor \sqrt{\ln X} \rfloor$ (and keep $A = D^2$), we obtain

$$S_3'(X) = X \sum_{d=1}^{D} \mu(d) \sideset{}{'}\sum_{a=3}^{A} \delta(d, a) + O\big(X D^{-3}\big).$$

Recalling that a above is restricted to odd values, we conclude that

$$\lim_{X \to \infty} X^{-1} S_3'(X) = \sum_{\substack{d=1 \\ (d,2)=1}}^{\infty} \mu(d) \sideset{}{'}\sum_{\substack{a=3 \\ (a,2)=1}}^{\infty} \delta(d, a). \tag{17}$$

Let $S_3''(X)$ be the part of $S_3'(X)$ that counts integers g subject to $g \equiv 55 \pmod{81}$. We can estimate $S_3''(X)$ using a variant of the above argument. It is not difficult to see that the conditions

$$d^4 \mid (g-1), \qquad g \equiv 55 \pmod{81}, \qquad g \equiv 1 - a_i^2 \pmod{a_i^2(a_i - 1)} \quad (1 \le i \le k),$$

are consistent if and only if $(d, 6) = 1$, the moduli $a_1, a_2, \dots, a_k$ satisfy (16) with $3d$ in place of d, and none of the a_i's is divisible by 9. However, since we are only interested in squarefree a_i's, the latter condition is superfluous. Thus, the argument leading to (17) also gives

$$\lim_{X \to \infty} X^{-1} S_3''(X) = \sum_{\substack{d=1 \\ (d,6)=1}}^{\infty} \mu(d) \sideset{}{'}\sum_{\substack{a=3 \\ (a,2)=1}}^{\infty} \delta(3d, a). \tag{18}$$

Finally, we note that the difference $S_3'(X) - S_3''(X)$ is exactly the counting function of $\mathcal{S}_3$. Hence, by (17) and (18), the set $\mathcal{S}_3$ has density

$$\delta_3 = \sum_{\substack{d=1 \\ (d,2)=1}}^{\infty} \mu(d) \sideset{}{'}\sum_{\substack{a=3 \\ (a,2)=1}}^{\infty} \delta(d, a) - \sum_{\substack{d=1 \\ (d,6)=1}}^{\infty} \mu(d) \sideset{}{'}\sum_{\substack{a=3 \\ (a,2)=1}}^{\infty} \delta(3d, a).$$

A simple calculation on a personal computer leads to the approximation $\delta_3 \approx .0564$.

3.3. **The density of $\mathcal{S}_4$.** The integers g that satisfy condition (iv) of Theorem 2 never satisfy condition (iii), for parity reasons, so we only need to exclude those g that satisfy conditions (i) or (ii). Consequently, the calculation of $\delta(\mathcal{S}_4)$ is very similar to that we just went through to calculate δ_3. Let $\mathcal{B}$ be the set of biquadrate-free values that the polynomial $f(a,b) = b^2a^2(a-1)$ takes when $a \equiv 3 \pmod 4$ and $b > 1$ is odd. Note that this restricts a and b to be squarefree and relatively prime. We reduce the calculation of the density of $\mathcal{S}_4$ to estimates for the distribution of sets of multiples of $\mathcal{B}$ in residue classes. The next lemma is a slight generalization of a classical result on the density of a set of multiples.

Lemma 5. *Let $\mathcal{B} = \{b_1, b_2, \dots\}$ be a set of positive integers such that $\sum_k b_k^{-1} < \infty$, and let a, q be positive integers. Then the set*

$$\mathcal{M}(\mathcal{B}; q, a) = \big\{n \in \mathbb{N} : n \equiv a \pmod q,\ n \text{ is divisible by some } b \in \mathcal{B}\big\}$$

has an asymptotic density given by

$$\sum_{k=1}^{\infty}\bigg(\frac{\epsilon(q,a;b_k)}{[q,b_k]} - \sum_{1\le j<k}\frac{\epsilon(q,a;b_k,b_j)}{[q,b_k,b_j]} + \sum_{1\le i<j<k}\frac{\epsilon(q,a;b_k,b_j,b_i)}{[q,b_k,b_j,b_i]} - \cdots\bigg),$$

where $\epsilon(q,a;b_k,b_j,\dots)$ is the indicator function of the condition $\gcd(q,[b_k,b_j,\dots]) \mid a$.

Proof. The case $q = 1$ is Theorem 9 in Halberstam and Roth [3, Ch. V], whose proof uses the inclusion-exclusion principle to count the elements of the union of the residue classes $x \equiv 0 \pmod{b_k}$. When $q > 1$, we use the Chinese Remainder Theorem to replace the latter union with the union of residue classes modulo $[q, b_k]$, defined by the conditions

$$x \equiv 0 \pmod{b_k}, \qquad x \equiv a \pmod q,$$

when those conditions are consistent (i.e., when $\epsilon(q,a;b_k) = 1$). We then follow the argument of Halberstam and Roth. □

We remark that the set $\mathcal{B}$ defined at the beginning of the section satisfies the hypothesis of Lemma 5. Indeed, if we denote the elements of $\mathcal{B}$ by $b_1, b_2, \dots$, we have

$$\sum_{k=1}^{\infty}\frac{1}{b_k} < \sum_{a=3}^{\infty}\sum_{b=3}^{\infty}\frac{1}{b^2a^2(a-1)} < \infty.$$

We may therefore apply Lemma 5 to sets $\mathcal{M}(\mathcal{B}; q, a)$ for various choices of q and a.

We have $\mathcal{S}_4 = \mathcal{S}_4' \setminus \mathcal{S}_4''$, where

$$\mathcal{S}_4' = \big\{h+1 : h \in \mathcal{M}(\mathcal{B}; 4, 2),\ h \text{ biquadrate-free}\big\},$$
$$\mathcal{S}_4'' = \big\{h+1 : h \in \mathcal{M}(\mathcal{B}; 324, 54),\ h \text{ biquadrate-free}\big\}.$$

Let $S_4'(X)$ and $S_4''(X)$ denote the counting functions of these two sets. Similarly to (14), we have

$$S_4'(X) = \sum_{\substack{d\le D\\ \gcd(d,2)=1}} \mu(d) \sum_{\substack{h\le X\\ h\in\mathcal{M}(\mathcal{B};4d^4,2d^4)}} 1 + O\big(XD^{-3}\big),$$

where D is a large integer. When X is sufficiently large in terms of D, we may use Lemma 5 to get

$$S_4'(X) = X \sum_{\substack{d\le D\\ \gcd(d,2)=1}} \mu(d)\beta(d) + O\big(XD^{-3}\big),$$

where

$$\beta(d) = \sum_{k=1}^{\infty}\bigg(\frac{\epsilon(4d^4,2d^4;b_k)}{[4d^4,b_k]} - \sum_{1\le j<k}\frac{\epsilon(4d^4,2d^4;b_k,b_j)}{[4d^4,b_k,b_j]} + \cdots\bigg).$$

Since $\mathcal{B}$ is contained in the residue class $x \equiv 2 \pmod 4$ and $d^4 \equiv 1 \pmod{16}$, we have

$$\frac{\epsilon(4d^4, 2d^4; b_k, b_j, \dots)}{[4d^4, b_k, b_j, \dots]} = \frac{1}{2[d^4, b_k, b_j, \dots]},$$

whence

$$\beta(d) = \frac{1}{2} \sum_{k=1}^{\infty} \bigg(\frac{1}{[d^4, b_k]} - \sum_{1 \le j < k} \frac{1}{[d^4, b_k, b_j]} + \cdots \bigg).$$

By letting $X \to \infty$ and then $D \to \infty$, we conclude that

$$\delta(\mathcal{S}_4') = \frac{1}{2} \sum_{\substack{d=1 \\ \gcd(d,2)=1}}^{\infty} \mu(d) \sum_{k=1}^{\infty} \bigg(\frac{1}{[d^4, b_k]} - \sum_{1 \le j < k} \frac{1}{[d^4, b_k, b_j]} + \cdots \bigg).$$

A similar argument can be applied to $\mathcal{S}_4''$ to show that

$$\delta(\mathcal{S}_4'') = \frac{1}{2} \sum_{\substack{d=1 \\ \gcd(d,6)=1}}^{\infty} \mu(d) \sum_{k=1}^{\infty} \bigg(\frac{\epsilon(81, 27; b_k)}{[81d^4, b_k]} - \sum_{1 \le j < k} \frac{\epsilon(81, 27; b_k, b_j)}{[81d^4, b_k, b_j]} + \cdots \bigg).$$

Note that $\epsilon(81, 27; b_k, b_j)$ is 0 or 1 according as 81 divides some or none of the integers $b_k, b_j, \dots$. Recalling that the elements of $\mathcal{B}$ are biquadrate-free, we deduce that

$$\delta(\mathcal{S}_4'') = \frac{1}{2} \sum_{\substack{d=1 \\ \gcd(d,6)=1}}^{\infty} \mu(d) \sum_{k=1}^{\infty} \bigg(\frac{1}{[81d^4, b_k]} - \sum_{1 \le j < k} \frac{1}{[81d^4, b_k, b_j]} + \cdots \bigg).$$

We conclude that the density of $\mathcal{S}_4$ is

$$\delta_4 = \delta(\mathcal{S}_4') - \delta(\mathcal{S}_4'').$$

Again, a computer calculation yields a numerical value of $\delta_4 \approx .0019$. This concludes the proof of Theorem 4.

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