Portraits of Groups on Bordered Surfaces

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Abstract

This paper looks at representing a group G as a group of transformations of an orientable compact bordered Klein surface. We construct visual representations (portraits) of three groups S_4, $Z_2 \times S_3$ and a group L^* of order 32. These groups have real genus 3, 2 and 5 respectively. The first two groups are M^*-groups; which means that they act on surfaces with maximal symmetry. They are also the only solvable M^* - simple groups.

1. Mathematical Background

The purpose of this paper is to construct visual representations (portraits) of finite groups on a compact orientable surface with one or more boundary components. A ***bordered Klein surface*** is a compact surface with one or more boundary components. The boundary components occur because an open disk is deleted from a surface. These will be referred to as ***holes***. This should be distinguished from the ***tunnel*** that is found in a torus. The surface that results after filling in all of the holes has the same topological genus as the bordered Klein surface with the holes. See [4] for more information. Each such surface is characterized by its topological genus, its orientability and the number of holes that it has. The introduction of the holes to a surface restricts us to a subgroup of the group of transformations of the surface without holes. We also want the surface to have as small a topological genus and as few holes as possible for the representation of the finite group.

The ***algebraic genus*** of a compact orientable surface X of topological genus p with k holes is given by $g(X) = 2p + k - 1$. In a certain sense, the algebraic genus measures the complexity of a topological surface. A finite group G acts on a surface X if G is isomorphic to a group of transformations of X. Given a finite group G, the ***real genus***, $\rho(G) = min\{g(X) | G \text{ acts on } X\}$ is the smallest integer in the set of integers that are the algebraic genus of some bordered surface on which G acts. Note that we are only considering bordered Klein surfaces here. Therefore, the real genus gives the algebraic genus of the "simplest" bordered Klein surfaces X on which the group G acts. Unfortunately, the algebraic genus does not uniquely describe the "simplest" surface. For example, a group of real genus five can conceivably act on sphere with six holes, a torus with four holes or a double torus with two holes. Knowing the real genus alone is not enough to distinguish between these three possibilities. Indeed, the group $Z_2 \times S_4$ acts on both the sphere with six holes and the torus with four holes.

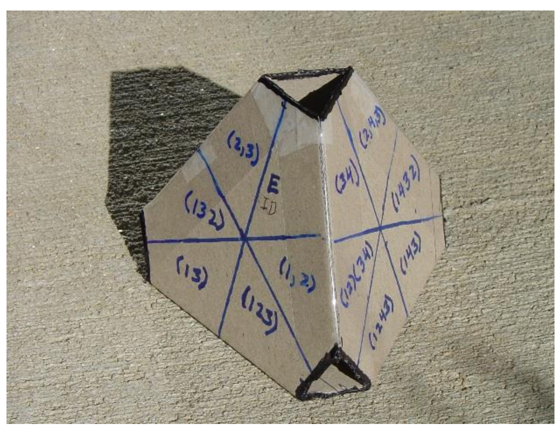

Figure 1: Action of S_4 on a tetrahedron with four holes

Finally, there is an analog of Hurwitz's Theorem. The order of the finite group G is less than or equal to

$12(\rho(G)-1)$. A group G with $|G| = 12(\rho(G)-1)$ is called an ***M* - group***. There is a connection between M^* - groups and regular maps (See [3] and [4]). A comprehensive reference for automorphism groups of bordered Klein surfaces is given in [2]. We have chosen to construct visual representations of the only two solvable M^* - groups and of a nilpotent group which is not an M^* - group.

2. Portrait of the Symmetric Group, S_4

The permutation group on 4 elements, S_4, is an M^* - group (see [4]), which acts on a sphere with four holes and so has real genus three. The group S_4 is also the group of symmetries of the tetrahedron. A tetrahedron is topologically a sphere with four distinguished points. The elements of S_4 act on the tetrahedron as either rotations, reflections or rotory reflections. Now if we cut a hole around each vertex of the tetrahedron and distort it into a sphere, we get a group of automorphisms of the sphere with four holes. A portrait of a group needs to tessellate the surface on which the group acts into regions where each region is the image of a fundamental region under the action of one of the group elements. A boundary edge must be taken to a boundary edge by each group element. It is easy to find the fundamental region for S_4 on a tetrahedron since each 2-cycle acts as a reflection through a plane. Each two cycle divides two of the four faces in half and interchanges the other two faces. Using all of the two cycles divides each face into six equal regions as shown in Figure 1. This gives a tessellation of the tetrahedron (or the tetrahedron with holes) and S_4 acts transitively on this tessellation. This means that there is a transformation, which maps each region into any other region. We designate one region as the identity, E, and we can label each region by the group element that takes E into that region. If we distort this tetrahedron with holes into a sphere with holes, and color the regions appropriately, we get a visual representation of how each element of S_4 moves the fundamental region. The resulting tessellation gives a visual representation of S_4.

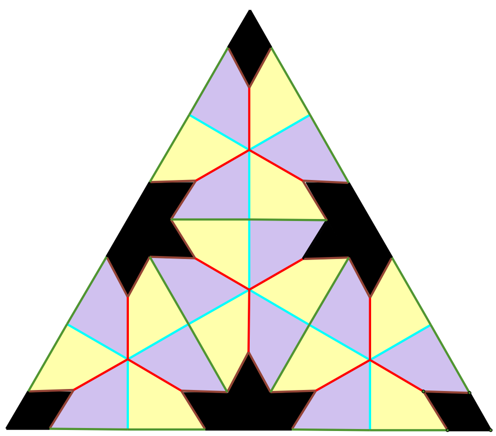

Figure 2: Polygonal representation of S_4

The method of finding a portrait given above is completely different from the one described by Burnside [1] for representing Riemann surfaces. Although Burnside didn't have the terminology of Non-Euclidean Crystallographic groups (NEC groups), the method described in Burnside [1] uses NEC groups to construct the portrait. An ***NEC group*** is a discrete subgroup of transformations of the hyperbolic plane. Any finite group G is the image of an NEC group under a homomorphism with no elements of finite order in its kernel (the **kernel** is the subgroup of all elements that map to the identity element in G). The surface can then be constructed as a quotient of the Poincare Disk. This method is described in Portraits of Groups [7]. Specifically, Burnside [1] uses inversions in a circle in the plane and then restricts to a disk to get the transformations of the Poincare Disk. Using 3 circles (2 circles and the circle of infinite radius) gives a fundamental region and constructs the group G as a quotient of the group $\Delta = \langle r, s, t \mid rst = 1\rangle$. A three sided region of the Poincare disk is labelled as the identity and colored white. Inversion in a side is represented by the generators r, s and t and the image of the identity under these transformations is colored black, since the orientation is reversed. Continuing this process results in a tessellation of the Poincare disk. Each region is labelled with the word representing the transformation that moves the identity region to the target region. Finally,

Figure 3: Portrait of S_4 from Figure 2

the compact surface that G acts upon is obtained by identifying regions corresponding to elements in the kernel of the map from Δ onto G. Variants of this method can be used with groups of higher rank (A Portrait of a Quadrilateral Group [8]). Klein surfaces can also be represented in this way (with slight differences). We can use an involution b for the border of a region with a hole. If that involution is then mapped to the identity in the quotient group, then there are no transformations that map a region into a hole. For example, a group G with maximal symmetry (an M^* - group) is the image of the NEC group with the following presentation.

$$(1) \qquad \Gamma = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = (da)^3 = 1 \rangle$$

The element b is contained in the kernel of the map onto G. This corresponds to a portrait of a quadrilateral group (See [8]), except that the generator that is contained in the kernel of the map corresponds to the border of the hole (ie. – there is no region of the surface on the other side of the generator b). The kernel of such a map is now called a ***bordered surface group*** since it contains only reflections, and no other elements of finite order. One interesting consequence of this is that every region must border one of the holes in these group visualizations. The quotient of Γ by the normal closure of $\langle b \rangle$ is given by $\bar{\Gamma} \cong \langle t, u, v \mid t^2 = u^2 = v^2 = (tu)^2 = (tv)^3 = 1 \rangle$. So $\bar{\Gamma}$ maps onto G. Therefore, as in [8], we can construct a polygonal representation of the group, where every face is a quadrilateral and the four edges of each face correspond to the generators a, b, c and d. Such a representation is given in Figure 2. In Figure 2 the black regions correspond to the holes, the yellow regions are images under an orientation preserving transformation and the purple regions are images under an orientation reversing action. Figure 2 folds up into Figure 3, which is the same portrait as in Figure 1. These two approaches are equivalent. Notice that because the order of (tu) in $\bar{\Gamma}$ is 2 and the order of (tv) in $\bar{\Gamma}$ is 3, this model has vertices of degree 4 and of degree 6. In fact, all M^* - groups must have portraits with vertices of degree 4 and vertices of degree 6. Also the order of the image of (uv) in G is 3 in this case. Therefore the hole has 6 faces bordering it. This is because the image of a, c and d in $\bar{\Gamma}$ is v, u and t respectively. Therefore, the edges adjacent to the edge bordering the hole correspond to u and v. It follows that the order of (uv) determines how many pairs of yellow and violet faces border the hole. Define the ***index*** or ***action index*** to be the order of (uv). In order to view these features, I have constructed a modified stereographic projection of the portrait of S_4 on the sphere with four holes. Figure 4 gives this projection. The outer rim is the hole at the top of the tetrahedron and the other three holes are the small white disks.

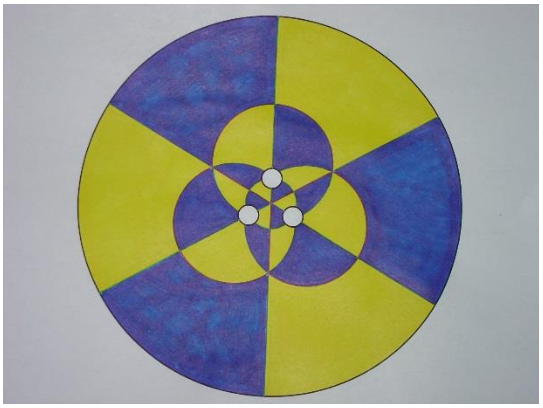

Figure 4: Stereographic projection of the portrait of S_4

Figure 5: $Z_2 \times S_3$ acting on the sphere

3. The group $Z_2 \times S_3$

The group $G = Z_2 \times S_3 \cong D_6$, where Z_2 is the cyclic group of order 2, S_3 is the permutation group on 3 elements and D_6 is the dihedral group of order 12. The group G has real genus 2. It acts on a sphere with three holes and on a torus with one hole [4]. It is easily seen that the action of G on the sphere with three holes is generated by 3 elements, $\boldsymbol{r}$, $\boldsymbol{a}$ and $\boldsymbol{t}$, where $\boldsymbol{r}$ is the reflection in the plane that contains

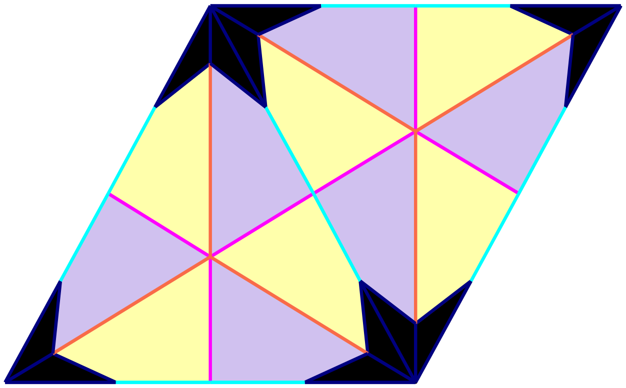

Figure 6: Polygonal representation of $Z_2 \times S_3$

the centers of the three holes, $\boldsymbol{a}$ is the 120 degree rotation about the line perpendicular to this plane and $\boldsymbol{t}$ is a 180 degree turn about a line through the center of the sphere and the center of one of the holes. The reflection $\boldsymbol{r}$ commutes with $\boldsymbol{a}$ and $\boldsymbol{t}$. Also the group $\langle \boldsymbol{a}, \boldsymbol{t} \rangle$ is clearly the only non-abelian group of order 6. The model of this is given in Figure 5, where a fundamental region if a pair of light and dark regions.

We are more interested in the action of $Z_2 \times S_3$ on the torus with one hole. This is the unique M^*- group which acts on the torus with one hole. The following construction is due to Greenleaf and May [4]. Define Λ to be the lattice subgroup of the complex numbers generated by $\{1, \omega\}$, where $\omega = e^{\pi i/3} = {(1 + i\sqrt{3})}/{2}$. The quotient surface $M = {\mathbb{C}}/{\Lambda}$ is the torus and if we delete a disk centered at the origin, we get a torus with one hole. The automorphisms of M defined by multiplication by ω and by complex conjugation, generate the group G which acts on the torus with one hole. The fundamental region of this lattice is the rhombus consisting of two equilateral triangles. It is fairly easy to see that, without the holes, the fundamental region for the action of $Z_2 \times S_3$ on M is the triangle with vertices at the origin, the centroid of the equilateral triangle and the base of an altitude. The hole then adds a fourth side by chopping off a small triangle near the origin. All of the 12 regions of the rhombus are the image under a unique group element of this fundamental region. The resulting rhombus is drawn in Figure 6 using Geometer's SketchPad. It is fun to draw the rhombus and the fundamental region first and then use the transformations in Geometer's SketchPad to tessellate the region. Since opposite sides of the rhombus are identified, we clearly have a torus as our topological surface. The group $G = Z_2 \times S_3$ is the image of the NEC group Γ, as defined by equation (1), by a bordered surface group. The action index (the order of uv) of this group is six. This means that all twelve faces border the hole. There are five vertices that are not on the boundary, three vertices of degree four and two of degree six. The two vertices that are at the centroid of the two equilateral triangles have degree six. The three vertices of degree two are at the midpoint of each edge of the equilateral triangles. There are only three, since each edge of one of the equilateral triangles is identified with a corresponding edge of the second. It is easiest to see the vertex of degree four at the centroid of the rhombus. Because opposite sides of the rhombus are identified, the other two vertices look the same. Figure 6 shows all of this clearly. Figure 7 shows the regions and the hole on an actual torus. The hole is on the outer rim and goes 5/6 of the way around the rim.

Figure 7: Model of $Z_2 \times S_3$ acting on the torus

4. An Example of Real Genus 5

Let $L^* = \langle T, U, V \mid T^2 = U^2 = V^2 = (TU)^2 = (TV)^4 = (UV)^8 = 1, (UV)^3 = TVUT \rangle$. The group L^* has order 32 and it is the group of automorphisms of the regular map $\{4,8\}_{1,1}$. A ***map*** is a decomposition of the surface into faces, edges and vertices. A map on a surface has an Euler characteristic, which depends only on the surface that the map is drawn upon. A map is ***regular*** if its automorphism group contains two automorphisms. One of these cyclically permutes the edges of a face, and the other cyclically permutes the edges that meet at a vertex. For more information, see Coxeter and Moser [3].

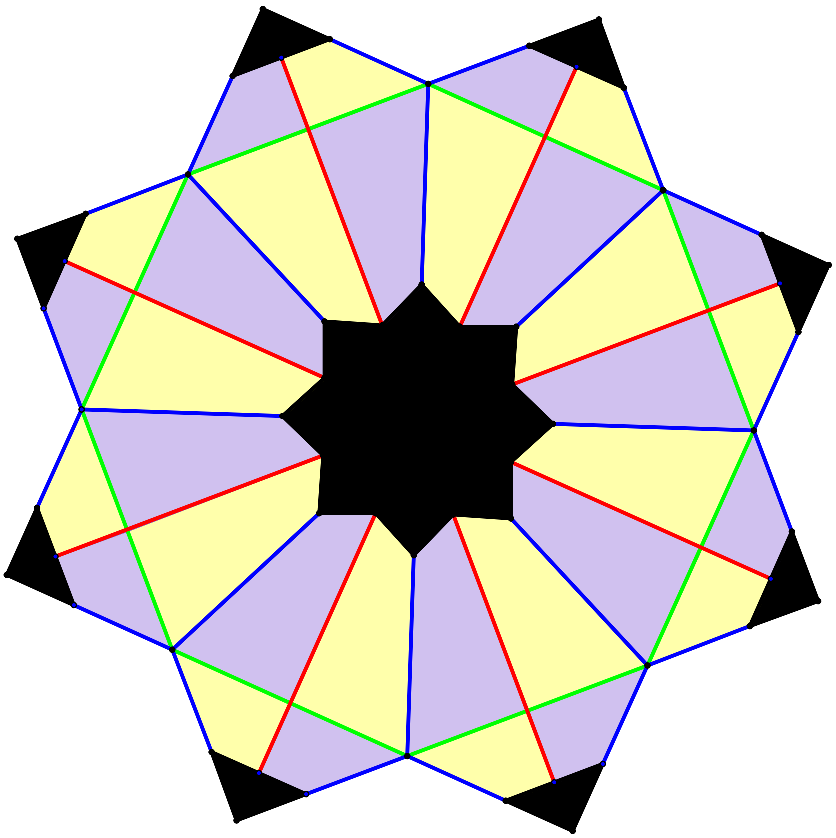

Figure 8: The polygonal representation for the real genus five surface

The group L^* acts on the double torus (two tunnels) with two holes and so it has real genus five [5]. Its order is 8(g-1) and so it is not an M^*- group. The regular map $\{4,8\}_{1,1}$ can be drawn on a double torus and computer graphics for this map exist [6]. The polygonal representation of this group is given in Figure 8 and Figure 9 gives a diagram of a model of this surface. The holes are on the inside of the tunnels in the double torus. There are 16 regions bordering each hole. The vertices of order 8 are on the opposite side of the tunnel from the hole. Finally, there are four vertices of order 4. If you use Figure 9 where the boundaries are filled with a disk with the obvious vertices and edges, you can check that the figure has 2 vertices of degree 16, 4 vertices of degree 8 and 8 vertices of degree 4. There are 32 faces (one for each group element) and 48 edges. This gives Euler characteristic -2 and fits on a genus 2 surface.

5. Conclusion

A specific finite group can be represented as a group of automorphisms of many different surfaces. The automorphisms used to represent the group can be restricted only to orientation preserving automorphisms or may include orientation reversing ones as well. In addition, the surface may have boundary components and this further restricts the types of automorphisms that represent the group. Finally, we may use surfaces of minimal complexity in some sense. If such a compact surface can be embedded in three dimensional space, then we may construct models of that surface. The images of the fundamental region under the action of the group elements tessellate the surface. Using a coloring scheme for the resulting tessellation results in interesting and often artistic models of these surfaces.

Sometimes the model is really interesting because of the high degree of symmetry in it. In other cases, it is the breaking of the symmetry that is most interesting. In the case of bordered Klein surfaces, the restriction that all regions must have one edge bounding a hole introduces some interesting breaks in the symmetry of the some of the models. Figure 7 is interesting because all regions must border the same hole and yet tessellate the torus minus the hole.

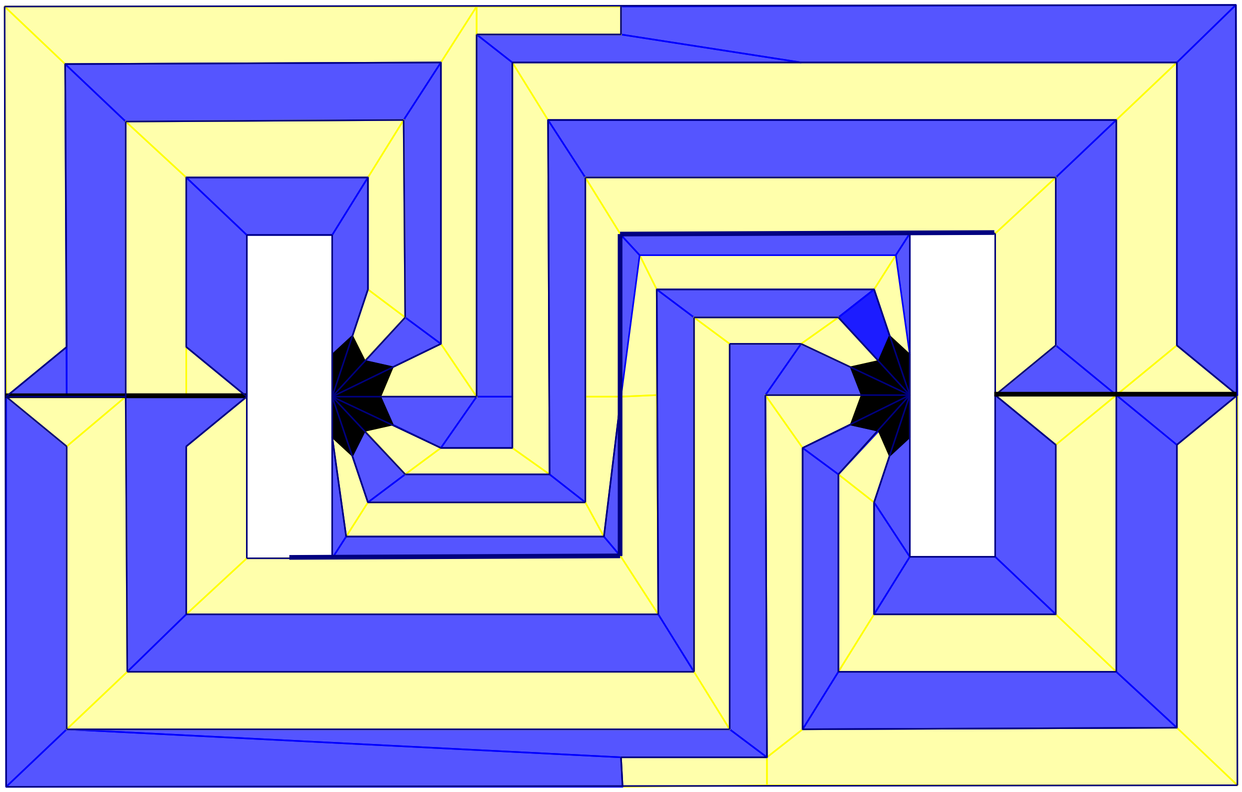

Figure 9: Diagram of L^* acting on a double torus

6. References

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