THE GENUS SPECTRUM OF FINITE ABELIAN GROUPS

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May, C.L. and Zimmerman, J., The Real Genus Spectrum of Abelian Groups, submitted Journal of Algebra and its Applications

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The *real genus* $\rho(G)$ is the minimum algebraic genus of any compact bordered Klein surface on which G acts faithfully.

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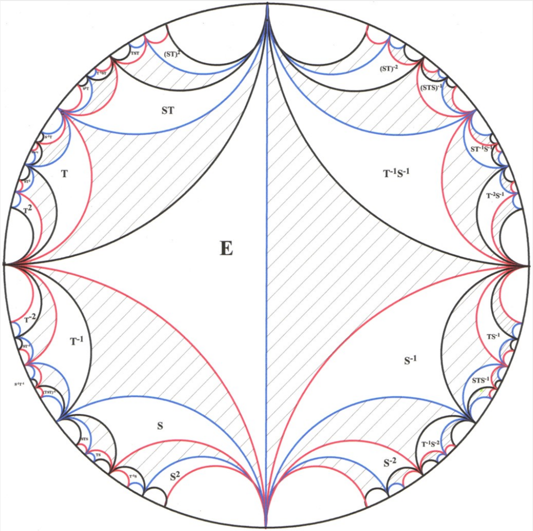
In the next picture, S, T, and U are reflections in a circle and these generators satisfy STU = 1. So $U = T^{-1}S^{-1}$ and S and T generate a free group.

History



E
ST
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S
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$ST^{-1}S^{-1}$
$T^{-2}S^{-1}$
S^{-1}
TS^{-1}
STS^{-1}
$T^{-1}S^{-2}$
S^{-2}

Burnside's Pictures

Surface Representation of the Quaternion Group

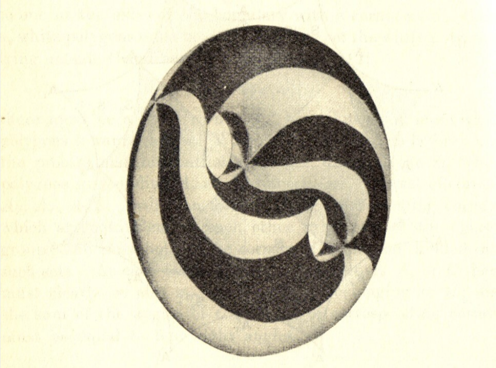

Fig. 8.

Pictures

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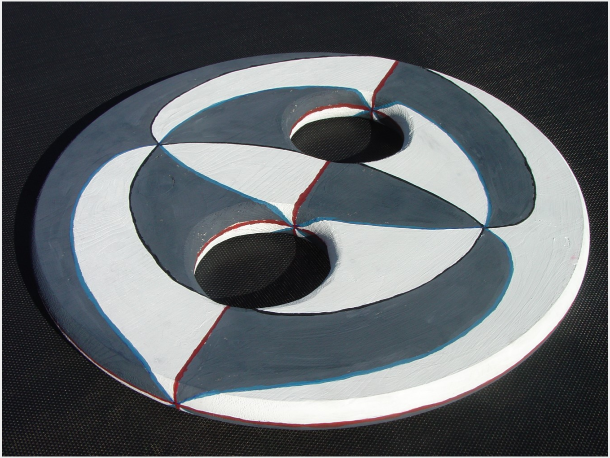
$G = \langle s, t | s^4 = t^4 = (st)^3 = 1, st = (ts)^2 \rangle$.

By looking at the number of vertices, edges and faces of this graph, you can see that the Euler characteristic is -2 and so the genus is 2.

My Models

Surface Representation of the Dicyclic Group



My Models

Surface Representation of a Group of Order 32

Procedure for determining the Genus

In order to determine the strong symmetric genus of a finite group G, you need to represent G as the quotient of a Fuchsian group Γ by a surface group K.

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A surface group is a Fuchsian group with no elements of finite order.

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Each such surface has a genus associated with it. The Fuchsian group has a "signature" associated with it and there is a formula which gives the genus of the Riemann surface for each signature.

Finally, the strong symmetric genus of the group G is the smallest genus from among the Riemann surfaces that G acts on preserving orientation.

Natural Questions

It is natural to try to compute the groups whose genus is a small positive integer.

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A well-known classical result of Hurwitz (1893) states that $|G| \leq 84(g-1)$ for genus g, where $g > 2$.

Therefore, there are only a finite number of groups whose genus is a positive integer greater than 2.
It is also natural to look at certain classes of groups with similar presentations and find a formula which gives the genus of each group in the class.

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The set of positive integers that is the strong symmetric genus of some group is called the strong symmetric genus spectrum.

Upper and Lower Density

Let A be a set of positive integers. For an integer X, let $[1, X]$ be the set of integers between 1 and X and define $A(X) = |A \cap [1, X]|$.

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If A is a set of integers, its *lower* and *upper asymptotic densities*, denoted $\underline{\delta}(A)$ and $\overline{\delta}(A)$, are given by

$$\underline{\delta}(A) = \liminf_{X \to \infty} \frac{A(X)}{X}$$

and

$$\overline{\delta}(A) = \limsup_{X \to \infty} \frac{A(X)}{X}.$$

Density in the Integers

A set A is said to have an *asymptotic density*, if $\underline{\delta}(A) = \overline{\delta}(A)$; when A does have an asymptotic density, it is denoted $\delta(A)$.

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Now let S be the set of all positive integers that are the strong symmetric genus of some finite group G. It follows that $\delta(S) = 1$.

Clearly the 2003 result of May and Zimmerman is considerably stronger than the above density statement.

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We will also give necessary and sufficient conditions for a positive integer g to be the strong symmetric genus of an abelian group.

Formulas I

Recall that every finite abelian group G has a canonical representation $G \cong Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}$, with standard invariants $m_1, m_2, \ldots, m_r$ subject to $m_1 > 1$ and $m_i | m_{i+1}$ for $1 \leq i < r$.

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Maclachlan (1965) proved that if G is an abelian group of rank $r \geq 3$, with $|G| \geq 10$, then

$$\sigma^0(G) = 1 + \frac{|G|}{2} \min_{0 \leq \gamma \leq r/2} \left\{ 2\gamma - 2 + \sum_{i=1}^{r-2\gamma} \left(1 - \frac{1}{m_i}\right) + \left(1 - \frac{1}{m_{r-2\gamma}}\right) \right\}. \tag{1}$$

Formulas II

For example, when $a > 1$ and $a^3 n \geq 10$, Maclachlan's formula yields

$$\sigma^0(\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}) = 1 - a^2 + a^2(a-1)n.$$

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In particular, when $a = 2$, this reveals that S contains the entire residue class $g \equiv 1 \pmod 4$.

Also when a is odd, $g \equiv 1 - a^2 \pmod a)^2(a-1)$ and this is equivalent to $g - 1$ is divisible by a^2 for some odd integer a with $(a-1)|g$ by the Chinese Remainder Theorem.

Formulas III

When $b \geq 2$ and $bn > 2$, Maclachlan's formula gives

$$\sigma^0(Z_a \times Z_{ab} \times Z_{abn}) = 1 + b^2 a^2 (a-1) n. \tag{2}$$

In all cases, except when a and b are odd, with $a \equiv 3 \pmod 4$, $g \equiv 1 \pmod 4$.

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Proposition

The spectrum of abelian groups of rank 3 *consists of the congruence class* $g \equiv 1 \pmod 4$ *and the integers* g *satisfying conditions (iii) or (iv) of the Theorem below.*

Main Theorem

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Let $g \geq 2$. Then $g \in S$ if and only if g satisfies one of the following conditions:

(i) $g \equiv 1 \pmod 4$ *or* $g \equiv 55 \pmod{81}$;

(ii) $g-1$ *is divisible by* p^4 *for some odd prime* p;

(iii) $g-1$ *is divisible by* a^2 *for some odd integer* a *with* $(a-1) \mid g$;

(iv) $g-1$ *is divisible by* $b^2a^2(a-1)$ *for some odd integers* $a, b > 1$, *with* $a \equiv 3 \pmod 4$.

Rank Four Abelian Groups

Proposition

The spectrum of abelian groups of rank 4 *is a subset of the integers satisfying conditions (i) or (ii) of the Main Theorem. Moreover, the spectrum of abelian groups of ranks* 3 *or* 4 *contains all the integers satisfying condition (ii) of the Main Theorem.*

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The spectrum of abelian groups of rank 4 is a subset of the integers satisfying conditions (i) or (ii) of the Main Theorem. Moreover, the spectrum of abelian groups of ranks 3 or 4 contains all the integers satisfying condition (ii) of the Main Theorem.

Proof: Notice that for an abelian group to have rank 4, it must have a subgroup isomorphic to Z_p^4 for some prime p.

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When $a \geq 5$, then $\sigma^0(A) = 1 + |A|$ for the rank 4 abelian group A. For $a = 3$, then $\sigma^0(A) = 1 + |A|$ or $\sigma^0(A) \equiv 1 \pmod 4$ for all except a few cases.

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For the exceptional cases with $a = 3$, $\sigma^0(A) \equiv 55 \pmod{81}$.

Conversely, all numbers g of the form $1 + p^4 n$ are the genus of groups of rank 3 or 4.

High Rank Abelian Groups

Let A be an abelian group of rank $n \geq 5$. So A has a subgroup isomorphic to Z_a^n. If a is even, then $\sigma^0(A) \equiv 1 \pmod 4$ and $\sigma^0(A) = \sigma^0(Z_2 \times Z_2 \times Z_{2n})$ for some n.

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If a is odd, then there is a rank four group B satisfying $|A| = |B|$ and so $\sigma^0(A) = \sigma^0(B)$.

Therefore, the genus spectrum is given by looking at the strong symmetric genus of groups of rank 3 or rank 4.

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$$\delta_1 = \frac{1}{4} + \frac{1}{81} - \frac{1}{324} = \frac{7}{27}.$$

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It is known that if $\gcd(a, q) = 1$, then

$$\sum_{\substack{n \leq X \\ n \equiv a \pmod q}} \alpha_4(n) = C_q X + O(X^{1/3}).$$

Number Theory

We use this to show that the number $T(X)$ of integers $g \in \mathcal{S}_2$ with $g \equiv 2 \pmod 4$ and $g \leq X$ is

$$T(X) = \frac{X}{4} - \sum_{\substack{h \leq X \\ h \equiv 1 \pmod 4}} \alpha_4(n) = \frac{X}{4}\left(1 - \frac{16}{15\zeta(4)}\right) + O(X^{1/3}).$$

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and so the density of $\mathcal{S}_2$ is

$$\delta_2 = \frac{20}{27} - \frac{79}{100\zeta(4)} \approx 0.0108.$$

Number Theory

$\mathcal{S}_3$ can be described as the set $\mathcal{A}$ of integers g such that

$$g \equiv 1 - a^2 \pmod{a^2(a-1)} \qquad (*)$$

for some odd $a > 1$.

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for some odd $a > 1$.

The number of such g that fail condition (ii) is

$$\begin{aligned} S_3'(X) &= \sum_{\substack{g \in \mathcal{A} \\ g \le X}} \alpha_4(g-1) + O(1) \\ &= \sum_{d \le D} \mu(d) \sum_{3 \le a \le D^2} \sum_{\substack{g \in \mathcal{A}(d,a) \\ g \le X}} 1 + O(XD^{-3}), \end{aligned}$$

where $D = \lfloor \sqrt{\ln X} \rfloor$, $\mathcal{A}(d,a)$ is the set of $g \in \mathcal{A}$ such that $d^4 \mid (g-1)$ and a is the least odd for which $(*)$ holds.

Number Theory

Using inclusion-exclusion, it is possible to estimate the density of $\mathcal{A}(d,a)$ and to prove that, as $X \to \infty$,

$$S_3'(X) = X \sum_{d \le D} \mu(d) \sum_{3 \le a \le D^2} \delta(d,a) + o(X),$$

where $\delta(d,a)$ is a certain arithmetic function.

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where $\delta(d, a)$ is a certain arithmetic function.

From this and another similar calculation, we deduce that

$$\delta_3 \approx 0.0564.$$

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Altogether, we have

$$\delta(\mathcal{S}) = \delta_1 + \cdots + \delta_4 \approx 0.3284.$$

The Real Genus

Every finite abelian group G has a representation $G \cong Z_{e_1} \times \cdots Z_{e_m} \times Z_{d_1} \times \cdots \times Z_{d_\ell} \times Z_2^n$, where e_i is a multiple of 4 $d_j \geq 3$ is odd and as we move from left to right all invariants divide the previous one.

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McCullough (1990) obtained the formulas for the real genus of abelian non-cyclic groups by using graph theoretic techniques. One of 4 such formulas is given below.

$$\rho(G) = 1 + |G|\left(n - 1 + \sum_{i=1}^{\ell}\left(1 - \frac{1}{d_i}\right) + \sum_{j=n+1}^{m}\left(1 - \frac{1}{e_j}\right)\right).$$

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In Maclachlan's formula the last term is repeated. This is not true in any of McCullough's formulas. This small change makes a huge difference. It changes where the minimum value for the summation occurs.

For example, this small change means that we must consider abelian groups of all ranks when looking at the real genus spectrum.

Basic Results

It is easy to show that

$$\rho(Z_2 \times Z_2 \times Z_{2c}) = 1 + 4c.$$

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Let A be a finite abelian group of even order with $\rho(A)$ positive. If $\rho(A)$ is not congruent to $1 \pmod 4$, then the Sylow 2-subgroup of A is cyclic.

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Theorem

Let $g \geq 4$. If $g \in \mathcal{S}$ and $g-1$ is squarefree, then $g = \rho(A)$ for an abelian group A of rank two.

Main Results

We also obtain the following necessary condition for an integer g to be in the spectrum.

Theorem

Let $g \geq 4$. If $g \in \mathcal{S}$, then g satisfies one of the following conditions:

(i) $g \equiv 1 \pmod 4$;

(ii) $g \equiv 4 \pmod 6$ *or* $g \equiv 16 \pmod{20}$;

(iii) $g - 1$ *is divisible by a for some odd integer a such that $(a-1)$ divides g;*

(iv) $g - 1$ *is divisible by p^2 for some odd prime p;*

Density

It is a classical result that the squarefree integers have asymptotic density $6/\pi^2 \approx .6079$. The squarefree integers are distributed among the three classes of integers congruent to $1, 2, 3 \pmod 4$. Further, Jameson proved that the asymptotic density of the odd squarefree integers is $4/\pi^2 \approx .4053$. Using these results, we have the following bounds for the density.

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$$0.4771 \leq \underline{\delta}(\mathcal{S}) \leq \overline{\delta}(\mathcal{S}) \leq .6010.$$

Density

It is a classical result that the squarefree integers have asymptotic density $6/\pi^2 \approx .6079$. The squarefree integers are distributed among the three classes of integers congruent to $1, 2, 3 \pmod 4$. Further, Jameson proved that the asymptotic density of the odd squarefree integers is $4/\pi^2 \approx .4053$. Using these results, we have the following bounds for the density.

$$0.4771 \leq \underline{\delta}(\mathcal{S}) \leq \overline{\delta}(\mathcal{S}) \leq .6010.$$

THE END