# GROUP ACTIONS ON SURFACES

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For the purposes of this talk, a discrete group acting on a surface is a set of one to one functions from the points on a compact surface in three dimensions onto itself. The composition of two functions in the set is another function in the set and the inverse of any function in the set is also in the set.

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Since a donut has 1 hole, this would be called a genus 1 action of $D_8$.

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The reflections reverse the orientation of the surface and the rotations preserve the orientation.

Unfortunately, there are a lot of finite groups which cannot act on a sphere or torus. Therefore, we have to figure out how to construct more complicated surfaces.

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The group elements will be represented by transformations of a plane. Then by identifying certain regions of the plane we will obtain a compact surface.

# Definitions

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There is a connected region of the surface, called the Fundamental Region, that is moved around the surface by the inversions.

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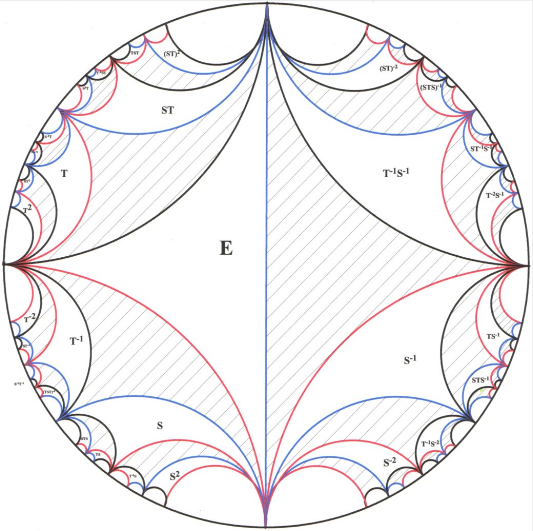
In the next slide, we will look at an infinite group with 2 generators acting on the Poincare disk model of Hyperbolic space.

The elements S, T, and U are reflections in a circle and these generators satisfy STU = 1. So $U = T^{-1}S^{-1}$ and S and T generate a free group.

# Background



E
ST
T
T^2
T^-2
T^-1
S
S^2
(ST)^2
(ST)^-2
(STS)^-1
T^-1S^-1
ST^-1S^-1
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TS^-1
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In the diagram on the Poincare disk, the regions containing words which realize the same group element are identified. This effectively "folds the hyperbolic plane into a compact surface."

# Definitions

The Quaternion Group $Q$ is a non-abelian group with 8 elements and it has presentation $Q = \langle R, S | R^4 = S^4 = 1, R^2 = S^2, R^{-1}SR = S^{-1} \rangle$.

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Therefore it has 24 edges.

# Burnside's Picture

Surface Representation of the Quaternion Group

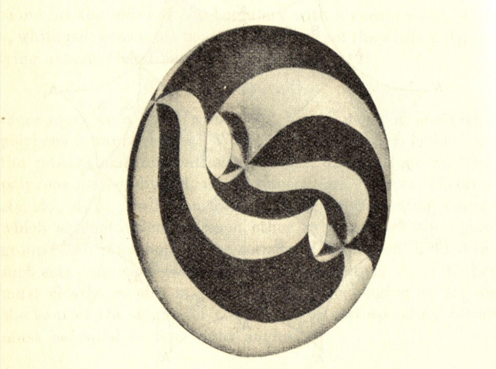

Fig. 8.

# Euler Characteristic

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The genus g of a surface is the number of "donut" holes that it has.

The Euler characteristic is related to the genus by $\chi = 2 - 2g$. So the surface for the quaternion group has genus 2.

# Pictures

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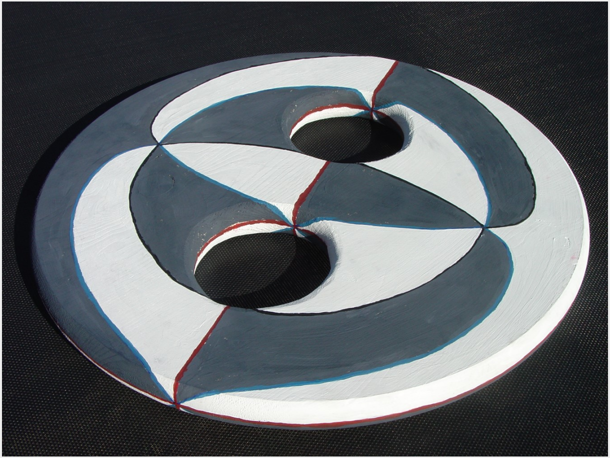
$G = \langle s, t | s^4 = t^4 = (st)^3 = 1, st = (ts)^2 \rangle$.

By looking at the number of vertices, edges and faces of this graph, you can see that the Euler characteristic is -2 and so the genus is 2.

# My Models

Surface Representation of the Dicyclic Group



# My Models

Quasiabelian Group of Order 16

# My Models

Surface Representation of a Group of Order 32

# Genus Parameters

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If we allow orientation reversing actions, we get the symmetric genus, $\sigma(G)$.

If we look at non-orientable or bordered surfaces, we get other genus parameters.

# Procedure for determining the Strong Symmetric Genus

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A surface group is a Fuchsian group with no elements of finite order.

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Each such surface has a genus associated with it. The Fuchsian group has a "signature" associated with it and there is a formula which gives the genus of the Riemann surface for each signature.

Finally, the strong symmetric genus of the group $G$ is the smallest genus from among the Riemann surfaces that $G$ acts on preserving orientation.

# Procedure for determining the Strong Symmetric Genus

It is relatively easy to find a homomorphism from a Fuchsian group onto a finite group $G$. This surjection induces an action of $G$ on a Riemann surface defined by taking a quotient of the upper half - plane (as a model of hyperbolic space) and using $G$ to identify regions.

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The genus depends on the non-Euclidean area of the fundamental region. The smaller the area, the smaller the genus.

# Procedure for determining the Strong Symmetric Genus

The hard part is to show that there is no homomorphism from a Fuchsian group whose fundamental region has smaller area onto $G$. Once you have the smallest possible area, you have the Strong Symmetric Genus.

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So to find $\sigma^0(G)$ for a finite group $G$, you must find a Fuchsian group and a homomorphism onto $G$ with kernel a surface group and the Fuchsian group must have the smallest possible non-Euclidean area for a Fundamental region.

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# Natural Questions

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Therefore, there are only a finite number of groups whose genus is a positive integer greater than 2.

It is also natural to look at certain classes of groups with similar presentations and find a formula which gives the genus of each group in the class.

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**May & Zimmerman, 1993.** In the paper, The symmetric genus of finite abelian groups, Dr. May and I updated MacLachlan's results to include orientation reversing actions and gave a formula for the symmetric genus of a finite abelian group.

# Natural Questions II

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This is hard because given a positive integer, it is not clear what finite group $G$ will have that integer as its genus.

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$\sigma^0(DC_n) = n$ for $n$ even.

$\sigma^0(\mathbb{Z}_k \times D_n)$ also covers all even positive integers for appropriate values of $k$ and $n$.

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**Conder and Tucker, 2011**. If $g$ is any non-negative integer such that $g$ is not congruent to 8 or 14 $\pmod{18}$, then there exists a finite group $G$ with symmetric genus $\sigma(G) = g$. Moreover, the same holds if $g \cong 8$ or $14 \pmod{18}$ and every factor $p^e$ in the prime-power factorization of $g - 1$ is congruent to 1 $\pmod{6}$.

# Small Numbers

**Ph.D. Thesis, Stephen Lo, 2018** If $g$ is a non-negative integer less than 1000, then $g = \sigma(G)$ for some finite group $G$, unless $|G| = 392, 536, 800, \mathbf{836}, \mathbf{980}$.

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Most researchers in this area believe and conjecture that there are no gaps in the symmetric genus spectrum.

It is also possible to look at the (strong) symmetric genus spectrum for classes of finite groups.

# Upper and Lower Density

Let $A$ be a set of positive integers. For an integer X, let $[1, X]$ be the set of integers between 1 and X and define $A(X) = |A \cap [1, X]|$.

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If $A$ is a set of integers, its *lower* and *upper asymptotic densities*, denoted $\underline{\delta}(A)$ and $\overline{\delta}(A)$, are given by

$$\underline{\delta}(A) = \liminf_{X \to \infty} \frac{A(X)}{X}$$

and

$$\overline{\delta}(A) = \limsup_{X \to \infty} \frac{A(X)}{X}.$$

# Density in the Integers

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Now let $S$ be the set of all positive integers that are the strong symmetric genus of some finite group $G$. It follows that $\delta(S) = 1$.

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Now let $S$ be the set of all positive integers that are the strong symmetric genus of some finite group $G$. It follows that $\delta(S) = 1$.

Clearly the 2003 result of May and Zimmerman is considerably stronger than the above density statement.

# Density of the Abelian Spectrum

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**Borror, Morris, Tarr, 2014.** If $J$ is the strong symmetric genus of all abelian groups, then $\delta(J) \geq \frac{643}{2025} > .3175$.

They did such a nice job that I had to wonder how much further we could go.

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We showed that $\delta(J)$ exists and that it is approximately .3284.

We also gave necessary and sufficient conditions for a positive integer $g$ to be the strong symmetric genus of an abelian group.

# Formulas III

The following theorems are from **Kumchev, A. V., May, C. L., Zimmerman, J. J. (2017). The strong symmetric genus spectrum of abelian groups. Archiv der Mathematik, 108(4), 341-350.**

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## Proposition

*The spectrum of abelian groups of rank* $3$ *consists of the congruence class* $g \equiv 1 \pmod 4$ *and the integers* $g$ *satisfying conditions (iii) or (iv) of the Theorem below.*

# Main Theorem

## Main Theorem

*Let $g \geq 2$. Then $g$ is the strong symmetric genus of an abelian group if and only if $g$ satisfies one of the following conditions:*

(i) $g \equiv 1 \pmod 4$ *or* $g \equiv 55 \pmod{81}$*;*

(ii) $g-1$ *is divisible by* $p^4$ *for some odd prime* $p$*;*

(iii) $g-1$ *is divisible by* $a^2$ *for some odd integer* $a$ *with* $(a-1) \mid g$*;*

(iv) $g-1$ *is divisible by* $b^2a^2(a-1)$ *for some odd integers* $a, b > 1$*, with* $a \equiv 3 \pmod 4$*.*

# Rank Four Abelian Groups

## Proposition

*The spectrum of abelian groups of rank* $4$ *is a subset of the integers satisfying conditions (i) or (ii) of the Main Theorem. Moreover, the spectrum of abelian groups of ranks* $3$ *or* $4$ *contains all the integers satisfying condition (ii) of the Main Theorem.*

# Rank Four Abelian Groups

## Proposition

*The spectrum of abelian groups of rank* 4 *is a subset of the integers satisfying conditions (i) or (ii) of the Main Theorem. Moreover, the spectrum of abelian groups of ranks* 3 *or* 4 *contains all the integers satisfying condition (ii) of the Main Theorem.*

Abelian groups of rank 5 or greater have the same genus as an abelian group of rank 3 or 4.

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If we allow orientation reversing actions, then $|G| \leq 168(g-1)$ and $G$ is called an extended Hurwitz group.

# Maximum Group Order

A well-known classical result of **Hurwitz (1893)** states that if a finite group $G$ acts **preserving orientation** on a surface of genus $g$ where $g > 2$, then $|G| \leq 84(g-1)$.

If $|G| = 84(g-1)$, then $G$ is called a Hurwitz group.

If we allow orientation reversing actions, then $|G| \leq 168(g-1)$ and $G$ is called an extended Hurwitz group.

Hurwitz groups and extended Hurwitz groups only occur for a small number of genera $g$. See **Conder, M. (2010). An update on Hurwitz groups.**

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There are well-known families of extended Hurwitz groups that provide an infinite number of integers $g$ satisfying $M(g) = 2N(g)$. It is also easy to see that there are solvable groups which provide an infinite number of such examples.

# Maximum Group Order

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**May and Zimmerman, 2022.** Surprisingly, there are an infinite number of positive integers $g$ where $M(g) = N(g)$.

# Maximum Group Order

**May and Zimmerman, 2022.** showed that if $p$ is a large enough prime number satisfying $p \equiv 1 \pmod 6$ and $g = 2p+1$ or $g = 3p+1$, then $M(g) = N(g) = 24(g-1)$.

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Suppose $p \equiv 5 \pmod 6$. We prove that if $p$ is also congruent modulo 25 to 1, 6, 11 or 16, then $N(g) = 8(g+11)$ and $M(g) = 16(g+11)$; otherwise $N(g) = 8(g+1)$ and $M(g) = 16(g+1)$.

# THE END

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