p-Adics and Valued Fields

A Graduate Reading Course on a Model-Theoretic View of Valued Fields
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Books: *p-adic Analysis Compared with Real* by Sveylana Katok
Model theory of valued fields Lecture Notes by Lou van den Dries

Introduction:

In this reading course we will learn about valued fields in general, and the *p*-adics in particular, from a model-theoretic standpoint. I will not assume the participant has much background in model theory; we will cover what is needed as we need it.

What are valued fields? They are simply fields with a valuation structure, which is a homomorphism from the multiplicative group of the field to an ordered abelian group satisfying the "ultrametric triangle inequality." A prime example of this are the *p*-adic numbers, $\mathbb{Q}_p$. To get these, we take the "*p*-adic valuation," $x \longmapsto v_p(x)$, on $\mathbb{Q}$ and take the completion (analogous to the completion of $\mathbb{Q}$ to $\mathbb{R}$). Here, $v_p(n)$ is the highest power of *p* that divides *n*, and then extend this to $\mathbb{Q}$.

Why are valued fields, and in particular, the *p*-adics, of interest? The *p*-adics play an integral role in the study of algebra, number theory, and analysis. By redefining what we mean by "close" (namely, how much *p* do numbers have in common), we get a new metric that can be used to encode number theoretic information, such as congruences.

What will we cover in the course? Well, first of all, I will need to explain in detail what I said in the preceding paragraphs. We will begin by discussing local rings, valuation rings, and the Henselian property. We will explore the topological properties of valued fields, including its "ultrametric" nature. We will discuss all of this vis-à-vis the *p*-adics, painting a picture of this seemingly mysterious algebraic and topological field. Finally, time permitting, we will discuss the Ax-Kochen properties of valued fields. Of particular interest is the following theorem:

The Ax-Kochen Principle. Let σ be any first-order sentence in the language of rings. Then

$$\mathbb{Z}_p \vDash \sigma \Leftrightarrow \mathbb{F}_p[\![t]\!] \vDash \sigma$$

for all but finitely many primes *p*.

That is, the *p*-adic integers "sort of look like" the formal power series over the finite field of *p* elements. Of course, this should be taken with a grain of salt, as σ must be a *first-order* property of rings. For example, the field of fractions of $\mathbb{Z}_p$ has characteristic zero whereas the field of fractions of $\mathbb{F}_p[\![t]\!]$ has characteristic *p*.

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