ON VC-MINIMAL FIELDS AND DP-SMALLNESS

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ABSTRACT. In this paper, we show that VC-minimal ordered fields are real closed. We introduce a notion, strictly between convexly orderable and dp-minimal, that we call dp-small, and show that this is enough to characterize many algebraic theories. For example, dp-small ordered groups are abelian divisible and dp-small ordered fields are real closed.

INTRODUCTION

Recently, model theorists have been working on using the progress in stability theory as a template for work in unstable theories. Since much of modern mathematics is done outside the stable world, it seems reasonable to explore such avenues.

The notion of a good "minimality" condition comes up frequently in stability theory, and there have been many useful suggestions for a suitable "minimality" condition in the unstable context. S. Shelah developed dp-minimality, which was subsequently studied extensively by many others [5, 8, 13, 15]. Another property, strictly stronger than dp-minimality, that was extensively studied is weak o-minimality [12]. In [1], H. Adler introduces the notion of VC-minimality, which sits strictly between weak o-minimality and dp-minimality. This too has been studied a great deal recently [4, 6, 7, 9]. In [9], this author and M. C. Laskowksi develop a new "minimality" notion called "convex orderability," which sits strictly between VC-minimality and dp-minimality.

When turned toward specific classes of theories, these minimality properties can yield strong classification results. For example, Theorem 5.1 of [12] asserts that every weakly o-minimal ordered group is abelian divisible and Theorem 5.3 of [12] says that every weakly o-minimal ordered field is real closed. For another example, Proposition 3.1 of [15] yields that every dp-minimal group is abelian by finite exponent and Proposition 3.3 of [15] states that every dp-minimal ordered group

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is abelian. In a similar spirit, J. Flenner and this author show, in [7], that all convexly orderable ordered groups are abelian divisible.

The goal of this paper is two-fold. In the first part of this paper, we introduce a new "minimality" condition that we call "dp-smallness," which fits strictly between convex orderability and dp-minimality. We then show that most of the results of [7] work when we replace "convexly orderable" with "dp-small." The second part of the paper is devoted to answering, in the affirmative, Open Question 3.7 of [7]. That is, we show that every convexly orderable ordered field is real closed. Stronger than that, we actually show this for dp-small ordered fields.

Notation. Throughout this paper, let T be a complete theory in a language L with monster model $\mathcal{U}$. We will use x, y, z, etc. to stand for tuples of variables (instead of the cumbersome $\overline{x}$ or $\vec{x}$). For any $A \subseteq \mathcal{U}$ and tuple x, let A_x denote the set of all $|x|$-tuples from A (so $A_x = A^{|x|}$). If $|x| = 1$, we will say that x is of the *home sort*.[1] For a formula $\varphi(x)$ and $A \subseteq \mathcal{U}$, let

$$\varphi(A) = \{a \in A_x : \mathcal{U} \models \varphi(a)\}.$$

For ordered groups G, let G_+ denote the set of positive elements of G. Similarly define F_+ for ordered fields F. For a dense ordered group G, let $\overline{G}$ denote the completion of G (in the sense of the ordering). For valued fields (F, v, Γ) (where $v : F^\times \to \Gamma$ is the valuation), for $a, b \in F$, let $a|b$ hold if and only if $v(a) \leq v(b)$.

Outline. In Section 1, we give all the relevant definitions and state the main results of the paper. We define dp-smallness in Definition 1.4. Theorem 1.6 shows that many of the results of [7] hold for dp-smallness instead of convex orderability. Finally, Theorem 1.7 states that all dp-small ordered fields are real closed, generalizing Theorem 5.3 of [12]. In Section 2, we provide a proof that dp-smallness does fit strictly between convex orderability and dp-minimality. We also prove Theorem 1.6. In Section 3, we prove Theorem 1.7. In Section 4, we discuss VC-minimal fields in general. We show that VC-minimal stable fields are algebraically closed and conjecture that all VC-minimal fields are either algebraically closed or real closed.

1. Definitions and Results

The following definition is due to H. Adler in [1].

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[1]We could also consider theories with multiple sorts, but for the purposes of this paper, we will need a single "home sort."

Definition 1.1. Fix a set of formulas $\Psi = \{\psi_i(x; y_i) : i \in I\}$ (where x is a free variable in every formula, but the y_i's may vary). We say that Ψ is *directed* if, for all $i, j \in I$, $b \in \mathcal{U}_{y_i}$, and $c \in \mathcal{U}_{y_j}$, we have that one of the following holds:

(1) $\models \forall x(\psi_i(x; b) \rightarrow \psi_j(x; c))$,
(2) $\models \forall x(\psi_j(x; c) \rightarrow \psi_i(x; b))$, or
(3) $\models \neg \exists x(\psi_i(x; b) \wedge \psi_j(x; c))$.

We say that a theory T is *VC-minimal* if there exists a directed $\Psi = \{\psi_i(x; y_i) : i \in I\}$ where x is of the home sort and each $L(\mathcal{U})$-formula $\theta(x)$ is a boolean combination of instances of formulas from Ψ.

This is an important concept because it generalizes some other "minimal" notions in model theory. For example, every strongly minimal theory is VC-minimal and every o-minimal theory is VC-minimal. Moreover, a few interesting algebraic examples are VC-minimal, including algebraically closed valued fields. In [7], J. Flenner and this author classify VC-minimality in certain algebraic structures using an intermediate tool called convex orderability. This notion was first introduced in [9].

Definition 1.2. An L-structure M is *convexly orderable* if there exists $\lhd$ a linear order on M (not necessarily definable) such that, for all L-formulas $\varphi(x; y)$ with x in the home sort, there exists $K_\varphi < \omega$ such that, for all $b \in M_y$, the set $\varphi(M; b)$ is a union of at most K_φ $\lhd$-convex subsets of M.

It is shown in [9] (Proposition 2.3) that convex orderability is an elementary property, so we say a theory T is *convexly orderable* if for any (equivalently all) $M \models T$, M is convexly orderable. It is also shown in [9] (Theorem 2.4) that all VC-minimal theories are convexly orderable. One reason convex orderability is preferable over VC-minimality is that it is easier to show a theory is not convexly orderable. This is the main tool used in obtaining the results from [7].

Theorem 1.3 (Main results of [7])**.** *The following hold:*

(1) *If $T = \mathrm{Th}(G; +, <)$ is the theory of an infinite ordered group, then T is convexly orderable if and only if G is abelian divisible.*
(2) *If $T = \mathrm{Th}(F; +, \cdot, <)$ is the theory of a convexly orderable ordered field, then all positive elements of F have an nth root for all $n < \omega$.*
(3) *If $T = \mathrm{Th}(A; +)$ is the theory of an abelian group, then T is convexly orderable if and only if T is dp-minimal and A has upward coherence (see Definition 2.4 below).*

(4) *If $T = \mathrm{Th}(F; +, \cdot, |)$ is the theory of an Henselian valued field that is convexly orderable, then Γ the value group is divisible.*

In each of the cases above, the crux of the argument is using a combinatorial property about the theory to show that it can not be convexly ordered. It boils down to the following notion.

Definition 1.4. We say that a partial type $\pi(x)$ is *dp-small* if there does not exist $\varphi_i(x)$ an $L(\mathcal{U})$-formula for $i < \omega$, $\psi(x; y)$ an L-formula, and $b_j \in \mathcal{U}_y$ for $j < \omega$ such that, for all $i_0, j_0 < \omega$, the type

$$\pi(x) \cup \{\varphi_{i_0}(x), \psi(x; b_{j_0})\} \cup \{\neg\varphi_i(x) : i \neq i_0\} \cup \{\neg\psi(x; b_j) : j \neq j_0\}$$

is consistent. We say T is *dp-small* if $x = x$ is dp-small where x is of the home sort.

Compare this to the definition of ICT-patterns and dp-minimality (Definition 2.1 below). In fact, dp-smallness implies dp-minimality (see Proposition 2.2 below).

As promised, we have the following relationship between dp-small and convexly orderable.

Proposition 1.5. *If T is convexly orderable, then T is dp-small.*

In this paper, we show that dp-smallness is enough to get all the results in Theorem 1.3. This generalizes most of the results from [7]. That is,

Theorem 1.6 (Results of [7], revisited). *The following hold:*
  (1) *If $T = \mathrm{Th}(G; +, <)$ is the theory of an infinite ordered group, then T is dp-small if and only if G is abelian divisible.*
  (2) *If $T = \mathrm{Th}(F; +, \cdot, <)$ is the theory of a dp-small ordered field, then all positive elements of F have an nth root for all $n < \omega$.*
  (3) *If $T = \mathrm{Th}(A; +)$ is the theory of an abelian group, then T is dp-small if and only if T is dp-minimal and A has upward coherence.*
  (4) *If $T = \mathrm{Th}(F; +, \cdot, |)$ is the theory of a dp-small Henselian valued field, then Γ the value group is divisible.*

Beyond this, the other main result of this paper is the following. This answers Open Question 3.7 of [7] in the affirmative.

Theorem 1.7. *Suppose that $\mathfrak{F} = (F; +, \cdot, <)$ is an ordered field and $T = \mathrm{Th}(\mathfrak{F})$. The following are equivalent.*
  (1) *T is VC-minimal.*
  (2) *T is convexly orderable.*
  (3) *T is dp-small.*

(4) $\mathfrak{F}$ *is real closed.*

As an immediate corollary, we get the following.

Corollary 1.8. *Suppose that* $\mathfrak{F} = (F; +, \cdot, <, |)$ *is a dp-small ordered Henselian valued field (with non-trivial valuation). Then* $\mathfrak{F}$ *is a real closed valued field.*

2. DP-Smallness

In this section, we prove Proposition 1.5 and Theorem 1.6.

Proof of Proposition 1.5. Suppose T is not dp-small. Therefore, there exists $L(\mathcal{U})$-formulas $\varphi_i(x)$ for $i < \omega$, an L-formula $\psi(x; y)$, and $b_j \in \mathcal{U}_y$ for $j < \omega$ (x is of the home sort) such that, for all $i_0, j_0 < \omega$,

$$\{\varphi_{i_0}(x), \psi(x; b_{j_0})\} \cup \{\neg\varphi_i(x) : i \neq i_0\} \cup \{\neg\psi(x; b_j) : j \neq j_0\}$$

is consistent. By replacing $\varphi_i(x)$ with $\varphi'_i(x) = \varphi_i(x) \wedge \bigwedge_{i' < i} \neg\varphi_{i'}(x)$, we may assume that the $\varphi_i(x)$ are pairwise inconsistent.

By means of contradiction, suppose T is convexly orderable. Say $\lhd$ is a convex ordering on $\mathcal{U}$. Further, let $K < \omega$ be such that, for all $b \in \mathcal{U}_y$, $\psi(\mathcal{U}; b)$ is a union of at most K $\lhd$-convex subsets of $\mathcal{U}$. Now look at $\varphi_i(x)$ for $i \leq 2K$ and suppose $L < \omega$ is such that, for all $i \leq 2K$, $\varphi_i(\mathcal{U})$ is a union of at most L $\lhd$-convex subsets of $\mathcal{U}$. Let $C_{i,\ell}$ enumerate these. That is, $C_{i,\ell} \subseteq \mathcal{U}$ is $\lhd$-convex and, for each $i \leq 2K$,

$$\varphi_i(\mathcal{U}) = \bigcup_{\ell < L} C_{i,\ell}.$$

By definition (and saturation of $\mathcal{U}$), for each $i \leq 2K$ and $j < \omega$,

$$\varphi_i(\mathcal{U}) \cap \psi(\mathcal{U}; b_j) \setminus \left(\bigcup_{j' \neq j} \psi(\mathcal{U}; b_{j'}) \right) \neq \emptyset.$$

By pigeon-hole, there exists $J \subseteq \omega$ infinite such that, for each $i \leq 2K$, there exists $\ell_i < L$ such that, for all $j \in J$,

$$C_{i,\ell_i} \cap \psi(\mathcal{U}; b_j) \setminus \left(\bigcup_{j' \neq j} \psi(\mathcal{U}; b_{j'}) \right) \neq \emptyset.$$

In particular, for any fixed $j \in J$, for all $i \leq 2K$,

$$C_{i,\ell_i} \cap \psi(\mathcal{U}; b_j) \neq \emptyset \text{ and } C_{i,\ell_i} \cap \neg\psi(\mathcal{U}; b_j) \neq \emptyset.$$

Without loss of generality, suppose $C_{0,\ell_0} \lhd C_{1,\ell_1} \lhd ... \lhd C_{2K,\ell_{2K}}$. For each even $i \leq 2K$, choose $a_i \in C_{i,\ell_i} \cap \psi(\mathcal{U}; b_j)$ and, for each odd $i \leq 2K$, choose $a_i \in C_{i,\ell_i} \cap \neg\psi(\mathcal{U}; b_j)$. Thus, $a_0 \lhd ... \lhd a_{2K}$ but it alternates

belonging to $\psi(\mathcal{U}; b_j)$. This contradicts the fact that $\psi(\mathcal{U}; b_j)$ is a union of at most $K \lhd$-convex subsets of $\mathcal{U}$. $\square$

To see that dp-smallness is, in fact, distinct from convex orderability, consider an example from [5]. Let L be the language consisting of unary predicates P_i for $i \in \omega_1$. Let T be the theory stating that, for each finite $I, J \subseteq \omega_1$ with $I \cap J = \emptyset$, there are infinitely many elements realizing

$$\bigwedge_{i \in I} P_i(x) \wedge \bigwedge_{j \in J} \neg P_j(x).$$

By Proposition 3.6 of [5], this theory is complete, has quantifier elimination, but is not VC-minimal. By Example 2.10 of [9], T is not convexly orderable. However, T is dp-small. To see this, notice that any supposed witness to non-dp-smallness would involve only countably many predicates P_i, but the reduct to countably many predicates is VC-minimal (see the discussion after Example 2.10 of [9]). How does dp-small compare to dp-minimality?

Definition 2.1. A partial type $\pi(x)$ is *dp-minimal* if there does not exist L-formulas $\varphi(x; y)$ and $\psi(x; z)$, $a_i \in \mathcal{U}_y$ for $i < \omega$, and $b_j \in \mathcal{U}_z$ for $j < \omega$ such that, for all $i_0, j_0 < \omega$, the type

$$\pi(x) \cup \{\varphi(x; a_{i_0}), \psi(x; b_{j_0})\} \cup \{\neg\varphi(x; a_i) : i \neq i_0\} \cup \{\neg\psi(x; b_j) : j \neq j_0\}$$

is consistent. We call such a witness to non-dp-minimality an *ICT-pattern*. We say T is *dp-minimal* if $x = x$ is dp-minimal where x is of the home sort.

Proposition 2.2. *If a partial type $\pi(x)$ is dp-small, then π is dp-minimal. In particular, all dp-small theories are dp-minimal.*

Proof. Notice that an ICT-pattern is, in particular, a witness to non-dp-smallness (where all the $\varphi_i(x)$ happen to be $\varphi(x; a_i)$). $\square$

Notice that Theorem 1.6 gives us examples of theories which are not dp-small but are dp-minimal. For example, the theory of Presburger arithmetic, $\mathrm{Th}(\mathbb{Z}; +, <)$, and the theory of the p-adics, $\mathrm{Th}(\mathbb{Q}_p; +, \cdot, |)$.

So we have the following picture, where each implication is strict:

VC-minimal $\Rightarrow$ convexly orderable $\Rightarrow$ dp-small $\Rightarrow$ dp-minimal.

We are now ready to prove Theorem 1.6. This basically involves minor tweaks to the proofs presented in [7], so we will skip some details here. First, we tackle the theory of ordered groups.

Proof of Theorem 1.6 (1). Let $T = \mathrm{Th}(G; \cdot, <)$ the theory of an ordered group. If G is abelian and divisible, then G is o-minimal, hence VC-minimal, hence convexly orderable, hence dp-small. Conversely,

suppose G is dp-small. By Proposition 2.2, it is dp-minimal, hence by Proposition 3.3 of [15], G is abelian. If G is not divisible, suppose we have a prime p so that $pG \neq G$. Check that the formulas

$$\varphi_i(x) = (p^i | x) \wedge (p^{i+1} \nmid x)$$

and $\psi(x; y, z) = y < x < z$ witness that T is not dp-small. This amounts to showing that $\varphi_i(\mathcal{U})$ is cofinal in G, which is Lemma 3.3 of [7]. $\qquad\square$

Notice that Theorem 1.6 (2) follows as an immediate corollary, since $(F_+; \cdot, <)$ is a dp-small ordered group, hence is divisible by Theorem 1.6 (1). We turn our attention to abelian groups. First, we recall some definitions from [7].

Let $T = \mathrm{Th}(A; +)$ for A some abelian group. For definable subgroups $B_0, B_1 \subseteq A$, define $\precsim$ a quasi-ordering as follows:

$$B_0 \precsim B_1 \text{ if and only if } [B_0 : B_0 \cap B_1] < \aleph_0.$$

This generates an equivalence relation $\sim$. Let $\mathrm{PP}(A)$ be the set of all p.p.-definable subgroups of A, which are finite intersections of subgroups of the form $\varphi_{k,m}(A)$ where

$$\varphi_{k,m}(x) = \exists y (k \cdot y = m \cdot x).$$

Let $\widetilde{\mathrm{PP}}(A) = \mathrm{PP}(A)/\sim$. Notice that $(\widetilde{\mathrm{PP}}(A); \precsim)$ is a partial order.

Proposition 2.3 (Corollary 4.12 of [3]). *The theory T is dp-minimal if and only if $(\widetilde{\mathrm{PP}}(A); \precsim)$ is a linear order.*

Definition 2.4 (Definition 5.9 of [7]). For $X \in \widetilde{\mathrm{PP}}(A)$, we say X is *upwardly coherent* if there exists $B \in X$ such that, for all $B_1 \in \mathrm{PP}(A)$ with $B \precsim B_1$, $B \subseteq B_1$. We say the group A is *upwardly coherent* if every $X \in \widetilde{\mathrm{PP}}(A)$ is upwardly coherent.

We are now ready to prove the next part of Theorem 1.6.

Proof of Theorem 1.6 (3). If T is dp-minimal and A is upwardly coherent, then by Theorem 5.11 of [7], T is convexly orderable, hence dp-small. Conversely, if T is dp-small, then T is dp-minimal by Proposition 2.2. So suppose that T is dp-minimal but A is not upwardly coherent. So fix $X \in \widetilde{\mathrm{PP}}(A)$ without upward coherence. Follow the construction in the proof of Theorem 5.11 of [7]. This gives us subgroups $B \in X$ and

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots \text{ from } \mathrm{PP}(A)$$

such that

(1) for each $i < \omega$, $A_i \cap B \neq A_{i+1} \cap B$ and

(2) for each $i < \omega$, $[A_i : A_i \cap B] \geq \aleph_0$.

Now it is easy to see that $\varphi_i(x) = [x \in (A_i \setminus A_{i+1})]$ and $\psi(x;y) = [(x - y) \in B]$ are a witness to non-dp-smallness. For more details, see the proof of Lemma 5.7 of [7]. $\qquad\square$

Finally, we turn our attention to Theorem 1.6 (4). However, as in [7], we will prove a more general result about simple interpretations.

Definition 2.5. Suppose M and N are structures in different languages, $A \subseteq M$, $S \subseteq M$ is A-definable, and $E \subseteq M \times M$ is an A-definable equivalence relation on S. We say that M *simply interprets* N over A if the elements of N are in bijection with S/E and the relations on S induced by the relations and functions on N via this bijection are A-definable in M.

Lemma 2.6. *If M simply interprets N over $\emptyset$ and M is dp-small, then N is dp-small.*

Proof. Let $\sigma : N \to S$ be the map given by simple interpretability (so σ induces a bijection from N to S/E). Suppose there exists $\varphi_i(x; z_i)$ for $i < \omega$ and $\psi(x;y)$ in the language of N, $a_i \in N_{z_i}$ for $i < \omega$, and $b_j \in N_y$ for $j < \omega$ witnessing non-dp-smallness. By definition, there exists $\varphi_i^*(x; z_i)$ for $i < \omega$ and $\psi^*(x;y)$ in the language of M corresponding to $\varphi_i(x; z_i)$ and $\psi(x;y)$ respectively. Then, one checks that $\varphi_i^*(x; \sigma(a_i))$ and $\psi^*(x; \sigma(b_j))$ witness to the fact that M is not dp-small. $\qquad\square$

Proof of Theorem 1.6 (4). Let $T = \mathrm{Th}(K; +, \cdot, |)$ a Henselian valued field. Check that the value group and residue field are simply interpretable in K. Then, by Lemma 2.6 and Theorem 1.6 (1), the value group is divisible. $\qquad\square$

Moreover, when we include the ordering in the language, by Lemma 2.6 and Theorem 1.7, the residue field is real closed. This, along with Theorem 1.6 (4), gives us Corollary 1.8. So, with this in mind, we switch gears and prove Theorem 1.7.

3. VC-Minimal Ordered Fields

In this section, we prove Theorem 1.7. In order to do this, we follow the proof of Theorem 5.3 in [12]. First, we give a trivial consequence of Theorem 3.6 of [15], but this formulation is useful to us here.

Lemma 3.1. *If $(G; <, +, ...)$ is a dp-minimal expansion of a divisible ordered abelian group and $X \subseteq G$ is definable, then X is the union of finitely many points and an open set.*

Proof. Let $\text{Int}(X) = \{a \in X : (\exists b, c \in X)[b < a < c \land (\forall x \in X)(b < x < c \to x \in X)]\}$ and let $\text{Ext}(X) = X \setminus \text{Int}(X)$. Then, since $\text{Ext}(X)$ is a definable subset of a dp-minimal expansion of a divisible ordered abelian group with no interior, by Theorem 3.6 of [15], $\text{Ext}(X)$ is finite. This gives the desired conclusion. $\square$

Next, we will need a result from [8].

Lemma 3.2 (Lemma 3.19 of [8]). *Let $\mathfrak{G} = (G; +, <, ...)$ be a dp-minimal expansion of a divisible ordered abelian group. Let $f : G \to \overline{G}_+$ be a definable function (to $\overline{G}_+$ the positive elements of the completion of G). Then, for each open interval $I \subseteq G$, there exists an open interval $J \subseteq I$ and $\epsilon > 0$ such that, for all $a \in J$, $f(a) > \epsilon$.*

The next ingredient is a slight modification of Theorem 3.2 of [8]. We need to be very careful, because this theorem as stated only works for definable functions to G and not necessarily to $\overline{G}$. However, a simple modification of the proof of Theorem 3.4 of [12] gives us the desired result.

Lemma 3.3. *Let $\mathfrak{G} = (G; +, <, ...)$ be a dp-minimal expansion of a divisible ordered abelian group and let $f : G \to \overline{G}$ be a definable function. Then, for any $b \in G$, there exists $\delta > 0$ such that, on the interval $(b, b + \delta)$, the function f is monotone.*

Proof. By Lemma 3.1, for each $x \in G$, one of the following holds:

(1) $\varphi_0(x) = (\exists x_1 > x)(\forall y)[x < y < x_1 \to f(x) < f(y)]$,
(2) $\varphi_1(x) = (\exists x_1 > x)(\forall y)[x < y < x_1 \to f(x) = f(y)]$, or
(3) $\varphi_2(x) = (\exists x_1 > x)(\forall y)[x < y < x_1 \to f(x) > f(y)]$.

Again by Lemma 3.1, there exists $\delta_1 > 0$ and $i = 0, 1, 2$ such that, for all $a \in (b, b + \delta_1)$, $\models \varphi_i(a)$. If $i = 1$, we are done. Without loss of generality, suppose $i = 0$ holds. Let

$$\chi(x) = (\forall x_1 > x)(\exists y, z)[x < y < z < x_1 \land f(y) \geq f(z)].$$

We show that $\chi((b, b + \delta_1))$ is finite. By Lemma 3.1, it suffices to show that $\chi((b, b + \delta_1))$ has no interior. So, suppose there exists an open interval $J \subseteq \chi((b, b + \delta_1))$. Define a function $g : J \to \overline{G}_+$,

$$g(x) = \sup\{z - x : z \in J, x < z, (\forall y \in (x, z))(f(x) < f(y))\}.$$

Since $\models \varphi_0(a)$ for all $a \in J$, $g(a) > 0$. Fix $\epsilon > 0$, $a, c \in J$ with $a < c$ and $c - a < \epsilon$. Since $\chi(a)$, there exists $d \in (a, b) \subseteq J$ such that $f(a) \geq f(d)$. Hence, $g(a) < \epsilon$. This contradicts Lemma 3.2.

Therefore, $\chi((b, b + \delta_1))$ is finite, hence there exists $\delta > 0$ such that f is strictly increasing on $(b, b + \delta)$. The proof works similarly when $i = 2$ (and we get f locally strictly decreasing after b). $\square$

The following theorem is a simple generalization of Lemma 3.1 (Theorem 3.6 of [15]) and Lemma 3.2 (Lemma 3.19 of [8]) respectively.

Theorem 3.4. *Let $\mathfrak{G} = (G; +, <, ...)$ be a dp-minimal expansion of a divisible ordered abelian group. Then, for each $n \geq 1$,*

- $(1)_n$ *If $X \subseteq G^n$ is a definable set with non-empty interior and $X = X_1 \cup ... \cup X_r$ is a definable partition of X, then, for some i, X_i has non-empty interior.*
- $(2)_n$ *If $f : G^n \to \overline{G}_+$ is a definable function, then for each open box $B \subseteq G^n$ there exists an open box $B' \subseteq B$ and $\epsilon > 0$ so that, for all $x \in B'$, $f(x) > \epsilon$.*

In particular, this holds for all dp-small expansions of an ordered group by Theorem 1.6 (1). This proof is easily adapted from the proof of Theorem 4.2 and Theorem 4.3 in [12].

Proof of Theorem 3.4. By simultaneous induction on n. Note that $(1)_1$ holds by Lemma 3.1 and $(2)_1$ holds by Lemma 3.2. Suppose $(1)_{n-1}$ and $(2)_{n-1}$ holds.

$(1)_n$: Let $X \subseteq G^n$ be a definable set with non-empty interior and $X = X_1 \cup ... \cup X_r$ be a definable partition of X. Fix $B \subseteq X$ an open box and let $\pi : G^n \to G^{n-1}$ be the projection onto the first $n - 1$ coordinates. Choose any $a \in G$ so that $(b, a) \in B$ for some $b \in G^{n-1}$. For each i, let

$$Z_i = \{b \in \pi(B) : (\exists a')(\forall x)(a < x < a' \to \langle b, x \rangle \in X_i)\}.$$

We claim that Z_i is a partition of $\pi(B)$. To see this, take $b \in \pi(B)$ and consider

$$Y_i = X_i|_b \cap (a, \infty) = \{x > a : \langle b, x \rangle \in X_i\}.$$

Since the Y_i form a partition of $X|_b \cap (a, \infty)$ which contains $B|_b$, by Lemma 3.1, there exists i and $a' > a$ so that $\{(b, x) : a < x < a'\} \subseteq X_i$. Hence $b \in Z_i$.

By $(1)_{n-1}$ on the Z_i, there exists an open box $B^* \subseteq \pi(B)$ and i so that $B^* \subseteq Z_i$. By replacing B with $(B^* \times G) \cap B$, we may assume that $\pi(B) \subseteq Z_i$. For each $b \in \pi(B)$, let $f(b)$ be the supremum of $a' - a$ for all a' as in the definition of Z_i. If $b \in (G^{n-1} \setminus \pi(B))$, set $f(b) = \infty$. Thus, clearly $f(b) > 0$ for all $b \in G^{n-1}$. By $(2)_{n-1}$, there exists $B' \subseteq \pi(B)$ and $\epsilon > 0$ so that, for all $x \in B'$, $f(x) > \epsilon$. Therefore, $B' \times (a, a + \epsilon) \subseteq X_i$ hence X_i has non-empty interior.

$(2)_n$: Let $f : G^n \to \overline{G}$ be a definable function and fix an open box $B \subseteq G^n$. For each $b = \langle b_1, ..., b_n \rangle \in B$ and each $j = 1, ..., n$, let $B_j = \{a \in G : \langle b_1, ..., b_{j-1}, a, b_{j+1}, ..., b_n \rangle \in B\}$, define a function

$g_{b,j} : B_j \to \overline{G}$ as follows:

$$g_{b,j}(a) = f(b_1, ..., b_{j-1}, a, b_{j+1}, ..., b_n).$$

By Lemma 3.3, there exists $\delta \in G_+$ so that, on $(b_j, b_j + \delta)$, the function $g_{b,j}$ is monotonic (either strictly increasing, strictly decreasing, or constant). As in the proof of Theorem 4.3 of [12], we first use $(1)_n$ on the type of monotonicity of $g_{b,j}$ after b_j for each j to reduce to a closed box B' such that, for each j, either

- $g_{b,j}$ is strictly increasing locally after b_j for all $b \in B'$,
- $g_{b,j}$ is strictly decreasing locally after b_j for all $b \in B'$, or
- $g_{b,j}$ is constant locally after b_j for all $b \in B'$.

Fix $c = \langle c_1, ..., c_n \rangle \in B'$. For each j, let π_j be the projection of G^n onto G^{n-1} by removing the jth coordinate, define a function $F_j : \pi_j(B') \to \overline{G}$ by setting $F_j(b_1, ..., b_{j-1}, b_{j+1}, ..., b_n)$ as the supremum of all $\delta \in G_+$ such that $g_{\langle b_1,...b_{j-1},c_j,b_{j+1},...,b_n \rangle, j}$ is monotone on the interval $(c_j, c_j + \delta)$. Hence, $F_j(b) > 0$ for all $b \in \pi(B')$. As in the proof of Theorem 4.3 of [12], use $(2)_{n-1}$ to shrink B' to an open box such that, for all j and $b \in B'$, $g_{b,j}$ is monotone on the projection of B' to the jth coordinate (the smallest corner is c).

Let $B' = I_1 \times ... \times I_n$. For each j, if $g_{b,j}$ is non-decreasing on I_j (for any equivalently all $b \in B'$), set k_j to be the left endpoint of I_j (i.e., $k_j = c_j$). Otherwise, set k_j to be the right endpoint of I_j. Notice that, for all $b \in B'$, $f(b) \geq f(k_1, ..., k_n) > 0$. This gives us the desired conclusion. $\square$

Corollary 3.5. *Let $\mathfrak{G} = (G; +, <, ...)$ be a dp-minimal expansion of a divisible ordered abelian group and fix $n \geq 1$. Suppose that B is an open box and $f, h : B \to \overline{G}$ are definable functions such that $h(x) < f(x)$ for all $x \in B$. If f or h is continuous, then $\{\langle x, y \rangle : x \in B, h(x) < y < f(x)\}$ has non-empty interior.*

Proof. Define $g : G^n \to G$ as follows

$$g(x) = \begin{cases} f(x) - h(x) & \text{if } x \in B, \\ \infty & \text{if } x \notin B. \end{cases}$$

Since $g(x) > 0$ for all $x \in G^n$, by Theorem 3.4 $(2)_n$, there exists an open box $B' \subseteq B$ and $\epsilon > 0$ so that, for all $x \in B'$, $g(x) > \epsilon$. Since the argument is symmetric, assume h is continuous. Choose any $a \in B'$. Since h is continuous, there exists an open box $B'' \subseteq B'$ containing a such that, for all $x \in B''$, $|h(x) - h(a)| < \epsilon/3$. It is easy to verify that $B'' \times (h(a) + \epsilon/3, h(a) + 2\epsilon/3)$ is an open box contained in $\{\langle x, y \rangle : x \in B, h(x) < y < f(x)\}$, as desired. $\square$

Following the outline of [12], we prove an analog of their Proposition 5.4 for dp-minimal ordered fields.

Proposition 3.6. *Let F be a dp-minimal ordered field with real closure R, fix $\alpha \in R$, and suppose that for each $\epsilon \in R$ with $\epsilon > 0$, there exists $b \in F$ such that $|\alpha - b| < \epsilon$. Then, $\alpha \in F$.*

This follows from the analog of Proposition 5.9 of [12] for dp-minimal ordered fields.

Proposition 3.7. *Fix F a dp-minimal ordered field. Let $p = \langle p_1, ..., p_n \rangle$ be an element of $F[x_1, ..., x_n]^n$ and $a \in F^n$ with $J_p(a) \neq 0$ (the Jacobian of p at a). Then, for every box $U \subseteq F^n$ containing a, the set $p(U)$ has non-empty interior.*

In order to prove this proposition, we use Lemma 5.5 and Corollary 5.8 of [12]. These are true of any ordered field F. We summarize these two in the following lemma.

Lemma 3.8. *Fix F any ordered field and let R be its real closure. Let $p \in F[x_1, ..., x_n]^n$ and let $B \subseteq R^n$ be any open box whose endpoints lie in F. Suppose that, for some $a \in (B \cap F^n)$, $J_p(a) \neq 0$. Then, there is an open box $U \subseteq B$ whose endpoint lie in F with $a \in U$ such that $p|_U$ is injective, $V := p(U)$ is open, and $p^{-1}|_{V \cap F^n}$ is continuous.*

Proof of Proposition 3.7. By induction on n. For $n = 1$, this follows from Lemma 3.1 (Theorem 3.6 of [15]).

Let $p \in F[x]^n$ and $a = \langle a_1, ..., a_n \rangle \in F^n$ with $J_p(a) \neq 0$ and fix $U = I_1 \times ... \times I_n \subseteq F^n$ an open box containing a. Since $J_p(a) \neq 0$, there exists some minor of the matrix $(\partial p_i / \partial x_j)_{i,j}$ has non-zero determinant. By swapping variables and functions, we may assume that $J_{p^*}(\pi(a)) \neq 0$, where $\pi : F^n \to F^{n-1}$ is the projection onto the first $n-1$ coordinates and $p^*(y) = \pi(p(y, a_n))$. By Lemma 3.8, there exists an open box $W \subseteq R^{n-1}$ with endpoints in F containing $\pi(a)$ with $W \cap F^{n-1} \subseteq \pi(U)$ satisfying $p^*|_W$ is injective, $p^*(W)$ is open, and $(p^*)^{-1}|_{p^*(W) \cap F^{n-1}}$ is continuous. The induction hypothesis says that $p^*(W \cap F^{n-1})$ has non-empty interior. Hence, there exists $U_0 \subseteq (W \cap F^{n-1})$ an open box in the sense of F so that $V_0 := p^*(U_0)$ is open (in F^{n-1}). Then, $p^*|_{U_0}$ is a homeomorphism between U_0 and V_0.

Notice that $U_0 \times I_n \subseteq U$ is an open box in the sense of F. Define

$$U_1 = \{y \in U_0 : (\exists z \in F) (\{p^*(y)\} \times (z, p_n(y, a_n)) \subseteq p(U_0 \times I_n))\}.$$
$$U_2 = \{y \in U_0 : (\exists z \in F) (\{p^*(y)\} \times (p_n(y, a_n), z) \subseteq p(U_0 \times I_n))\}.$$
$$U_3 = U_0 \setminus (U_1 \cup U_2).$$

By Theorem 3.4 $(1)_{n-1}$, we may assume that $U_0 = U_i$ for $i = 1, 2,$ or 3.

If $U_0 = U_1$, set $g(y)$ to the infimum of all z witnessing the condition of U_1. Using the functions $p_n(y, a_n)$ and $g(y)$ in Corollary 3.5, we get an open box $W \subseteq U_1$. This is the desired conclusion. A similar argument shows that, if $U_0 = U_2$, then U_0 has non-empty interior.

So suppose $U_0 = U_3$. Consider the definable set $Y = p_n(U_0 \times I_n)$. Clearly $p_n(b, a_n) \in Y$ for all $b \in U_0$. However, since $U_0 = U_3$, there are elements z arbitrarily close to $p_n(b, a_n)$ for which $z \notin Y$. By Lemma 3.1, there is an open interval I around $p_n(b, a_n)$ such that $I \cap Y = \{p_n(b, a_n)\}$. Define $h_1, h_2 : V_0 \to \overline{F}$ to be such that $(h_1(p^*(b)), h_2(p^*(b)))$ is the largest convex set witnessing this. By Corollary 3.5, there is an open box

$$W \subseteq \{\langle p^*(y), z \rangle : y \in U_0, h_1(p^*(y)) < z < h_2(p^*(y))\}.$$

By continuity, $p^{-1}(W) \cap F^n$ is open. However, $p^{-1}(W) \cap [U_0 \times I_n] \subseteq U_0 \times \{a_n\}$. Hence, it is not open. Contradiction. $\square$

The following lemma is contained in the proof of Proposition 5.4 in [12] and works for any field. The proofs of (1), (2), and (3) follow from the proof of Theorem 1 in [11]. See the proof of Proposition 5.4 in [12] for more details.

Lemma 3.9. *Let F a field and $\alpha \notin F$ algebraic over F. Let $\alpha = \alpha_1, ..., \alpha_n$ be the conjugates of α over F and let*

$$g(x_1, ..., x_n, y) = \prod_{i=1}^{n} \left(y - \sum_{j=0}^{n-1} \alpha_i^j x_j \right) = \sum_{j=0}^{n-1} G_j(x_1, ..., x_n) y^j + y^n.$$

Then,

(1) *$G_j(x) \in F[x]$ for all j.*
(2) *For $a \in F^n$, if $a_j \neq 0$ for some j, then $g(a, y)$ has no roots in F.*
(3) *There is some $d \in F^n$ such that $J_G(d) \neq 0$ and $d_j \neq 0$ for some j.*

Proof of Proposition 3.6. Fix F a dp-minimal ordered field and let R be its real closure. Suppose, by means of contradiction, that there exists $\alpha \in (R \setminus F)$ arbitrarily close to F. Construct $G = \langle G_1, ..., G_n \rangle$ as in Lemma 3.9 above for this choice of α, so conditions (1) through (3) hold (say (3) holds for some $d \in F^n$). By Proposition 3.7, there exists an open $U \subseteq F^n$ with $d \in U$ so that $V := G(U)$ has non-empty interior. By choosing U sufficiently small, we may assume that, for all $c \in U$, $J_G(c) \neq 0$ and $c_j \neq 0$ for some j. By Proposition 3.8, we may assume that $G|_U$ is a homeomorphism from U to V. Take $B \subseteq V$ an

open box and, without loss of generality, suppose $e := G(d) \in B$. Let $f : (R \setminus \{0\}) \to R$ be the function

$$f(y) = -y^n/y^{n-1} - \sum_{i=0}^{n-2} e_i y^i/y^{n-1}$$

and let $h(y) = \sum_{i=0}^{n-1} d_i y^i$. Note that $h(\alpha) \neq 0$ since $h(\alpha)$ is a root of $g(d, y)$, which has no roots in F by Lemma 3.9 (2). It is not hard to show that $f(h(\alpha)) = e_{n-1}$ (see the proof of Proposition 5.4 in [12] for more details). Hence, as $b \to \alpha$, $f(h(b)) \to e_{n-1}$. Therefore, there is $b \in F$ so that $h(b) \neq 0$ and

$$\langle e_0, e_1, ..., e_{n-2}, f(h(b)) \rangle \in B \subseteq V.$$

Since $G|_U$ is a homeomorphism, there exists $c \in U$ so that $G(c) = \langle e_0, e_1, ..., e_{n-2}, f(h(b)) \rangle$. So $c_j \neq 0$ for some j, hence by Lemma 3.9 (2), $g(c, y)$ has no roots in F. However, clearly $h(b)$ is a root of $g(c, y)$ in F, a contradiction. $\qquad\square$

Here is where we must part ways with dp-minimality and impose the stronger condition of dp-smallness. The main obstruction of using dp-minimality here is that dp-minimal ordered groups need not be divisible (for example, $(\mathbb{Z}, +, <)$). For the remainder of this section, suppose that $\mathfrak{F} = (F; +, \cdot, <)$ is a dp-small ordered field. Put on F the archimedean valuation $v : F \to \Gamma$, where $v(a) \geq 0$ for $a \in F$ if and only if there exists $n \in \mathbb{N}$ such that $|a| < n$. Let $\overline{F}$ be the residue field, V the valuation ring (i.e., the convex hull of the prime field), and M the maximal ideal in V (i.e., the set of infinitesimals near zero). So $\overline{F} = V/M$ is archimedean. We now follow the remainder of the proof in [12].

Lemma 3.10. *The value group Γ is divisible.*

Proof. By Theorem 1.6 (2), $(F_+; \cdot, <)$ is divisible. Hence, Γ is also divisible. $\qquad\square$

Lemma 3.11. *The residue field $\overline{F}$ is real closed.*

Proof. Suppose $\overline{F}$ is not real closed and let $p(x) \in V[x]$ be such that $\overline{p}(x)$ changes sign in $\overline{F}$ but has no root in $\overline{F}$. Then, define

$$M^* = \{a \in F : (\forall b \in F)(|p(b)| > |a|)\}.$$

It is not hard to show that $M^* = M$, the set of infinitesimals of F (for more details, see the proof of Proposition 5.11 of [12]). Therefore, v is definable. By Lemma 2.6, $(\overline{F}; +, \cdot, 0, 1)$ is dp-small (and, in particular, dp-minimal). Therefore, since $\overline{F}$ is archimedean, by Proposition 3.6, $\overline{F}$ is real closed. $\qquad\square$

Lemma 3.12. *The valued field (F, v, Γ) is Henselian.*

Proof. Suppose not. Then there exists a polynomial

$$p(x) = x^n + ax^{n-1} + \sum_{i=0}^{n-2} c_i x^i \in V[x]$$

with $v(a) = 0$ and $v(c_i) > 0$ for $i < n - 1$ that has no root in F. Moreover, there is $\alpha \in R$ with $p(\alpha) = 0$, $v(a - \alpha) > 0$, and $v(p'(\alpha)) = 0$ (for details, see Theorem 4 of [14]). Let

$$S := \{v(b - \alpha) : b \in F, v(b - \alpha) > 0\},$$

let S^* be the convex subgroup of Γ generated by S, and let $I := \{b \in F : v(b) > S^*\}$.

First, we show that S is cofinal in S^*. To see this, take $v(b - \alpha) \in S$ and let $c = b - p(b)/p'(b)$ $(v(b - \alpha) > 0$, so $v(p'(b)) = v(p'(\alpha)) = 0$, so $p'(b) \neq 0)$. Then, it is easy to check that $v(c - \alpha) \geq 2v(b - \alpha)$ using Taylor's Theorem. This shows that S is cofinal in S^*. Moreover, I is definable in $\mathfrak{F}$. To see this, notice that $v(a - \alpha) > 0$, hence $v(p(a)) = v(a - \alpha)$ by Taylor's Theorem (since $v(p'(\alpha)) = 0$). By Lemma 3.10, Γ is divisible, hence there exists $d \in F$ so that $v(d) = v(p(a))/2 = v(a - \alpha)/2$. If $\alpha > a$, set $c = a + d$ and otherwise set $c = a - d$. Therefore, α lies between a and c and, for all $b \in F$ between a and c, $v(b - \alpha) > 0$. Set J to be the interval in F between a and c. Hence, $I = \{d \in F : (\forall b \in J)(|p(b)| > |d|)\}$.

The valuation $v_0 : F \to \Gamma/S^*$ is definable, so look at the residue field F_0. As in the proof of Lemma 3.11, $\overline{\alpha} \in F_0$ (the image of α in R_0, the residue field of R with respect to v_0). Moreover, $\overline{\alpha}$ is in the convex hull of $\overline{J}$ in R_0, hence $\overline{\alpha} \in \overline{J}$. However, for all $b \in J$, $v(p(b)) = v(b - \alpha) \in S^*$. Therefore, $\overline{p}$ has no root in $\overline{J}$. We see that $\overline{\alpha}$ directly contradicts this fact. $\square$

Proof of Theorem 1.7. Notice that $(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3)$ is trivial. So we need only show $(3) \Rightarrow (4)$. Suppose $\mathfrak{F} = (F; +, \cdot, <)$ is a dp-small ordered field. By Lemma 3.10, Lemma 3.11, and Lemma 3.12, (F, v, Γ) is a Henselian valued field with a divisible value group and a real closed residue field. This implies that F itself is real closed. $\square$

4. VC-Minimal Fields

In this section we move away from ordered fields and discuss VC-minimal fields in general. We exhibit what is known about stable VC-minimal fields and state a conjecture about the nature of VC-minimal fields in general.

Remark 4.1. The theory of algebraically closed fields is VC-minimal. In particular, it is the reduct of ACVF, which is certainly VC-minimal.

In fact, piecing together results from [10] and [13], we get a much stronger result.

Theorem 4.2. *Let* $\mathfrak{F} = (F; +, \cdot)$ *be a field, let* $T = \text{Th}(\mathfrak{F})$, *and suppose* T *is stable and dp-minimal. Then,* $\mathfrak{F}$ *is algebraically closed.*

Proof. By Theorem 3.5 (iii) of [13], any theory T that is stable and dp-minimal has weight 1. By Corollary 2.4 of [10], stable and strongly dependent (in particular, dp-minimal) fields with weight 1 are algebraically closed.   □

In particular, the theory of separably closed fields with positive Eršov-invariant is not dp-minimal, or even strongly dependent. This is not surprising, since this theory is not even superstable [16].

We get, as an immediate corollary, the following fact about Henselian valued fields.

Corollary 4.3. *Suppose that* $\mathfrak{F} = (F; +, \cdot, |)$ *is a dp-small Henselian valued field (with non-trivial valuation) with a stable residue field. Then* $\mathfrak{F}$ *is an algebraically closed valued field.*

We have, in particular, that all stable VC-minimal fields are algebraically closed. On the other hand, by Theorem 1.7, all VC-minimal ordered fields are real closed. We also know that all VC-minimal unstable theories interpret an infinite linear order (Theorem 3.5 of [9]). This gives evidence for the following conjecture.

Conjecture 4.4 (VC-minimal fields conjecture)**.** *Let* $\mathfrak{F} = (F; +, \cdot)$ *be a field and let* $T = \text{Th}(\mathfrak{F})$. *Then* T *is VC-minimal if and only if* $\mathfrak{F}$ *is real closed or algebraically closed.*

This would, in turn, give a nice characterization of VC-minimal Henselian valued fields.

The gap in proving this conjecture is going from an infinite interpretable linear order (on a definable subset of F^n for n possibly much larger than 1), to a field ordering on F.

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