ON VC-DENSITY IN VC-MINIMAL THEORIES

VINCENT GUINGONA

Abstract. We show that any formula with two free variables in a VC-minimal theory has VC-codensity at most two. Modifying the argument slightly, we give a new proof of the fact that, in a VC-minimal theory where $\mathrm{acl}^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}$, the VC-codensity of a formula is at most the number of free variables (from [2, 9]).

1. Introduction

There is a strong connection between the study of NIP theories from model theory and the study of Vapnik-Chervonenkis dimension and density from probability theory. Indeed, as first noted in [12], a theory has NIP if and only if all definable families of sets have finite VC-dimension. Moreover, a definable family of sets has finite VC-dimension if and only if it has finite VC-density. Although VC-dimension provides a reasonable measure of the "complexity" of a definable set system in an NIP theory, it is highly susceptible to "local effects." Indeed a theory that is relatively "tame" globally but locally codes the power set of a large finite set will have high VC-dimension. On the other hand, VC-density is, in some respect, a much more natural measurement of complexity, impervious to such local complexity. Moreover, it is closely related to other measurements of complexity in NIP theories, most notably, the dp-rank (see, for example, [2, 8, 11]).

In the pair of VC-density papers by M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson, and S. Starchenko [2, 3], significant progress was made toward understanding VC-density in some NIP theories. Bounds were given for VC-density in weakly o-minimal theories, strongly minimal theories, the theory of the p-adics, the theory of algebraically closed valued fields, and the theory of abelian groups. However, many questions were left open. Perhaps the most interesting is the relationship between dp-rank and VC-density.

2010 *Mathematics Subject Classification.* 03C45.

Key words and phrases. VC-density, VC-minimal.

Thanks to: Anton Bobkov, Aaron Crighton, Alfred Dolich, M. Chris Laskowski, Sergei Starchenko, and the anonymous referee.

Open Question 1.1. *In a theory T, is it true that a partial type $\pi(y)$ has dp-rank $\leq n$ if and only if every formula $\varphi(x;y)$ has VC-density $\leq n$ with respect to $\pi(y)$?*

A simpler question, implied by this and the subadditivity of the dp-rank [11], is the following:

Open Question 1.2. *If T is dp-minimal and $\varphi(x;y)$ is any formula, then does φ have VC-density $\leq |y|$?*

Both of these appear to be very difficult questions to answer. So, we can ask an ostensibly easier question, replacing dp-minimality by something stronger.

VC-minimality was first introduced by H. Adler in [1]. A theory is VC-minimal if all definable families of sets in one dimension are "generated" by a collection of definable sets with VC-codimension ≤ 1. It turns out that all VC-minimal theories are indeed dp-minimal. Moreover, due to the close relationship between VC-dimension and VC-density, something can be said about VC-density in VC-minimal theories, to some degree. However, the primary question on computing VC-density in VC-minimal theories is still open.

Open Question 1.3. *If T is VC-minimal and $\varphi(x;y)$ is any formula, then does φ have VC-density $\leq |y|$?*

This question has been answered positively in the specific VC-minimal theory of algebraically closed valued fields [4]. It remains open for VC-minimal theories in general.

In this paper, we provide partial solutions to Question 1.3. The primary result is the following, which says this holds when $|y| \leq 2$.

Theorem 1.4. *If T is VC-minimal and $\varphi(x;y)$ is any formula with $|y| \leq 2$, then φ has VC-density ≤ 2.*

Although this theorem seems quite distant from answering Open Question 1.3, the proof is unique, employing a combinatorial method for dealing with directed systems, and may be of independent interest. For example, we will discuss using the method to provide a new proof for the weakly o-minimal case in [2].

Unfortunately, in general, the proof of Theorem 1.4 does not provide any bound on the VC-density of $\varphi(x;y)$ when $|y| > 2$. Some bounds could be obtained if there were uniform bounds on the number of components needed (independent of φ), but this is not a practical constraint.

2. VC-Codensity and Directedness

2.1. **VC-codensity.** Fix T a complete first-order theory in a language L with monster model $\mathcal{U}$. If x is a tuple of variables, let $|x|$ denote the length of x and let $\mathcal{U}_x$ denote the set $\mathcal{U}^{|x|}$ (more generally, if L is multisorted, we let $\mathcal{U}_x$ be the elements in $\mathcal{U}$ of the same sort as x).

If $\Phi(x;y) := \{\varphi_i(x;y) : i \in I\}$ is a set of formulas and $B \subseteq \mathcal{U}_y$, let $S_\Phi(B)$ be the Φ-*type space over* B. That is, $S_\Phi(B)$ is the set of all maximal consistent subsets of

$$\{\varphi_i(x;b)^t : b \in B, i \in I, t < 2\}.$$

Here we use the standard notation $\theta(x)^1 = \theta(x)$ and $\theta(x)^0 = \neg\theta(x)$ for formulas $\theta(x)$. Moreover, if P is an expression that can either be true or false, then we will denote $\theta^P = \theta$ if P is true and $\theta^P = \neg\theta$ if P is false. There exists a injective function $\eta : S_\Phi(B) \to {}^{B\times\Phi}2$ (i.e., into functions from $B \times \Phi$ to $\{0,1\}$) given as follows: For each $p \in S_\Phi(B)$, for each $b \in B$, for each $\varphi \in \Phi$, let

$$\eta(p)(b,\varphi) = \begin{cases} 1 & \text{if } p(x) \vdash \varphi(x;b), \\ 0 & \text{otherwise} \end{cases}.$$

Since η is injective, we obtain the bound $|S_\Phi(B)| \leq 2^{|B|\cdot|\Phi|}$. However, in interesting cases (i.e., when Φ has NIP), there is a polynomial bound instead of an exponential one. This leads to the following definition.

Definition 2.1 (VC-codensity)**.** Given a finite set of formulas $\Phi(x;y)$, the *VC-codensity* of Φ, denoted $\mathrm{vc}^*(\Phi)$, is the infimum over all $\ell \in \mathbb{R}$ such that there exists $K < \omega$ such that, for all finite $B \subseteq \mathcal{U}_y$,

$$|S_\Phi(B)| \leq K \cdot |B|^\ell.$$

If no such ℓ exists, say the VC-codensity is infinite ($\mathrm{vc}^*(\Phi) = \infty$).

The *VC-density* of finite set of formulas $\Phi(x;y)$ is the VC-codensity of $\Phi^{\mathrm{opp}}(y;x) := \Phi(x;y)$ (when we exchange the parametrization). We will only consider VC-codensity in this paper.

Consider the function $\pi_T : \omega \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, that we call the *VC-codensity function*, defined by

$$\pi_T(n) := \sup\{\mathrm{vc}^*(\Phi) : \Phi(x;y) \text{ is finite with } |x| = n\}.$$

Notice that $\pi_T(n) \geq n$. This is witnessed by the single formula

$$\varphi(x_0, \ldots, x_{n-1}; y) = \bigvee_{i<n} x_i = y.$$

As usual, if $\Phi = \{\varphi\}$ is a singleton, then $S_\varphi(B) = S_\Phi(B)$ and $\mathrm{vc}^*(\varphi) = \mathrm{vc}^*(\Phi)$. By coding tricks, it suffices to assume that the Φ in the definition of $\pi_T(n)$ are all singletons.

Lemma 2.2 (Sauer-Shelah Lemma)**.** *The following are equivalent for a formula $\varphi(x; y)$.*

(1) *φ has NIP,*
(2) *$\mathrm{vc}^*(\varphi)$ is finite.*

Even if a theory T has NIP, this does not guarantee that $\pi_T(n)$ is finite. Indeed we may have formulas $\varphi(x; y)$ with $|x| = 1$ and have $\mathrm{vc}^*(\varphi)$ arbitrarily large, even in the stable context. For example, countably many independent equivalence relations.

On the other hand, many interesting theories T have some bound on $\pi_T(n)$. For example, any weakly o-minimal theory T has $\pi_T(n) = n$ for all $n < \omega$ (Theorem 6.1 of [2]). The theory of the p-adics T has $\pi_T(n) \leq 2n-1$ (Theorem 1.2 of [2]). The theory of algebraically closed valued fields T has $\pi_T(n) = n$ (Corollary 1 of [4]).

The primary problem in the study of VC-codensity for theories is to determine when we can bound $\pi_T(n)$. For example, what conditions on T guarantee that $\pi_T(n) = n$?

2.2. **VC-minimaility.** A *quasi-forest* is a quasi-order $(F; \trianglelefteq)$ such that, for all $b, c \in F$, if there exists $a \in F$ such that $b \trianglelefteq a$ and $c \trianglelefteq a$, then $b \trianglelefteq c$ or $c \trianglelefteq b$. We say a quasi-forest $(F; \trianglelefteq)$ is a *quasi-tree* if there exists $a \in F$ such that, for all $b \in F$, $a \trianglelefteq b$. We drop the prefix "quasi" if $\trianglelefteq$ is a partial order.

For a set X and a set system on X, $\mathcal{C} \subseteq \mathcal{P}(X)$, we say that $\mathcal{C}$ is *directed* if, for all $A, B \in \mathcal{C}$, one of the following holds:

- $A \subseteq B$,
- $B \subseteq A$, or
- $A \cap B = \emptyset$.

Note that, if $\mathcal{C}$ is directed, then $(\mathcal{C} \setminus \{\emptyset\}; \supseteq)$ is a forest (and, if $X \in \mathcal{C}$, then it is a tree with root X).

In general, we can convert from formulas to set systems. If $\theta(x)$ is a formula (possibly with parameters), then let $\theta(\mathcal{U}) := \{a \in \mathcal{U}_x : \models \theta(a)\}$.

Fix $\Delta = \{\delta_i(x; y_i) : i \in I\}$ a set of partitioned formulas (where y_i is allowed to vary but x is fixed and $|x| = 1$) and consider the set system on $\mathcal{U}_x$,

$$\mathcal{C}_\Delta := \{\delta_i(\mathcal{U}; b) : i \in I, b \in \mathcal{U}_{y_i}\}$$

Definition 2.3 (Directedness)**.** We say Δ is *directed* if $\mathcal{C}_\Delta$ is directed. A formula $\delta(x; y)$ is *directed* if $\{\delta(x; y)\}$ is directed. An *instance* of Δ is a formula of the form $\delta_i(x; b)$ for some $i \in I$ and $b \in \mathcal{U}_{y_i}$.

Definition 2.4 (VC-minimality, [1])**.** A theory T is *VC-minimal* if there exists a directed set of formulas Δ (with $|x| = 1$) such that all formulas (with parameters) $\theta(x)$ are T-equivalent to a boolean combination of instances of Δ.

We will call the family Δ the *generating family* and we will call instances of Δ *balls*.

Remark 2.5 (VC-minimal versus fully VC-minimal). Suppose that T is VC-minimal with generating family Δ and let $\theta(x) = \varphi(x; b)$, where $\varphi(x; b)$ is an formula (without parameters) and $b \in \mathcal{U}_y$. Then, even though $\theta(x)$ is T-equivalent to a boolean combination of formulas of the form $\delta_i(x; c)$ for $c \in \mathcal{U}_{y_i}$, the parameters c are *a priori* unrelated to the parameter b. Much of the difficulty in counting φ-types comes from understanding the relationship between the parameters for φ and the parameters for the δ_i. If all formulas $\varphi(x; y)$ are T-equivalent to a boolean combination of formulas from Δ in the variables $(x; y)$ (i.e., the parameters are identical), we say that T is *fully VC-minimal.* See Definition 3.1 below.

Throughout the remainder of this paper, we assume T is a VC-minimal theory with generating family Δ.

Definition 2.6 (Unpackable, [5])**.** A directed family Δ is *unpackable* if no instance of Δ is T-equivalent to a disjunction of finitely many proper instances.

The following is a fundamental decomposition theorem for formulas in VC-minimal theories.

Theorem 2.7 (Theorem 4.1 of [6])**.** *For all formulas $\varphi(x; y)$ (with $|x| = 1$), there exists a directed formula $\delta(x; z)$, $N < \omega$, and formulas $\psi_i(x; y)$ for $i < 2N$ such that*

- *for all $b \in \mathcal{U}_y$, there exists $n \leq N$ and $c_i, \ldots, c_{n-1} \in \mathcal{U}_z$, $\psi_i(x; b)$ is T-equivalent to $\bigvee_{i<n} \delta(x; c_i)$, and*
- *for all $b \in \mathcal{U}_y$, $\varphi(x; b)$ is T-equivalent to*

$$\bigvee_{i<N} \psi_{2i}(x; b) \wedge \neg\psi_{2i+1}(x; b).$$

Remark 2.8 (Finite VC-minimality and u-balls). Throughout this paper, we will only be working with local properties of a VC-minimal theory (e.g., computing VC-codensity). In light of Theorem 2.7, we may assume that the generating family Δ is a singleton, $\{\delta\}$. Moreover, for the purposes of counting φ-types, it suffices to count $\{\psi_i(x; y) : i < 2N\}$-types (for ψ_i as in Theorem 2.7), so we can "replace" $\varphi(x; y)$

with these $\psi_i(x;y)$. Thus, we may assume that, for the formula $\varphi(x;y)$ considered, there exists $N < \omega$ such that, for all $b \in \mathcal{U}_y$, $\varphi(x;b)$ is T-equivalent to a disjunction of at most N instances δ. In [5], these are called *u-balls*.

Remark 2.9 (VC-minimality when acl = dcl). If $\mathrm{acl}^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}$ in T, then we may actually assume that all formulas are balls. For example, suppose $\varphi(x;y)$ is a formula and $N < \omega$ are such that, for all $b \in \mathcal{U}_y$, there exists $n \leq N$, $c_0, \dots, c_{n-1} \in \mathcal{U}_z$ such that,

$$\models (\forall x)\left(\varphi(x;b) \leftrightarrow \bigvee_{i<n} \delta(x;c_i)\right).$$

Then, in particular, $c_i/\delta \in \mathrm{acl}^{\mathrm{eq}}(b)$, hence $c_i/\delta \in \mathrm{dcl}^{\mathrm{eq}}(b)$. Thus, there exist formulas $\delta_i(x;y)$ for $i < N$ such that

- $\{\delta_i(x;y) : i < N\}$ is directed, and
- $\varphi(x;y)$ is T-equivalent to $\bigvee_{i<n} \delta_i(x;y)$.

Note that these directed δ_i now share the parameter variables with φ. For more details, see Section 3 below.

Now for any set $C \subseteq \mathcal{U}_z$ and any directed set $\Delta(x;z)$, there is a quasi-forest structure on $C \times \Delta$. Namely,

$$\langle c_0, \delta_0 \rangle \trianglelefteq \langle c_1, \delta_1 \rangle \text{ if } \models (\forall x)(\delta_1(x;c_1) \to \delta_0(x;c_0)).$$

This is a quasi-forest instead of a true forest because we could have that $\delta_0(x;c_0)$ and $\delta_1(x;c_1)$ are unequal but T-equivalent. Let $\mathcal{F}(C,\Delta) := (C \times \Delta; \trianglelefteq)$ denote this quasi-forest. We can expand $C \times \Delta$ by a "root," call it 0, and set $0 \trianglelefteq \langle c, \delta \rangle$ for all $c \in C$ (and $\langle c, \delta \rangle \trianglelefteq 0$ if $\models (\forall x)\delta(x;c)$). Then, $\mathcal{T}(C,\Delta) := (C \times \Delta \cup \{0\}; \trianglelefteq)$ is a quasi-tree.

Remark 2.10. Suppose C and Δ are finite. Each type in the Δ-type space $S_\Delta(C)$ corresponds to a node in the quasi-tree $\mathcal{T}(C,\Delta)$ (and, if Δ is unpackable, this is a bijective correspondence). To see this, for each $c \in C$ and $\delta \in \Delta$, consider the "generic Δ-virtual type" corresponding to the (interior) of the ball $\delta(x;c)$, namely $\nu_{c,\delta} : C \times \Delta \to 2$ is given by, for all $c' \in C$ and $\delta' \in \Delta$,

$$\nu_{c,\delta}(c',\delta') := \begin{cases} 1 & \text{if } \models (\forall x)(\delta(x;c) \to \delta'(x;c')), \\ 0 & \text{otherwise} \end{cases}$$

and let $\nu_0 : C \times \Delta \to 2$ be the constantly zero function, the generic virtual type of the root (needed if no ball is the whole space yet balls do not cover the whole space). Define the *virtual type space*

$$\mathcal{V}_\Delta(C) := \{\nu_{c,\delta} : c \in C, \delta \in \Delta\} \cup \{\nu_0\}.$$

By the directedness of Δ, it is not hard to see that, $\eta(S_\Delta(C)) \subseteq \mathcal{V}_\Delta(C)$. Note, however, that if Δ is packable, then this may be a proper inclusion. If a ball is the union of proper subballs, then the generic type corresponding to this ball is inconsistent. This is why we refer to these as *virtual* types.

It is necessary to consider only finite C and Δ. For example, in the theory of dense linear orders, if $C = \mathbb{Q}$ and $\Delta = \{x < y\}$, then $S_\Delta(C)$ has size $2^{\aleph_0}$. On the other hand, as defined, clearly $\mathcal{V}_\Delta(C)$ is countable. Indeed, $\mathcal{V}_\Delta(C)$ misses all non-proper cuts.

In particular we get that, for finite C and finite Δ,

$$|S_\Delta(C)| \leq |\Delta| \cdot |C| + 1.$$

Thus, for a VC-minimal theory T,

$$\pi_T(1) = 1.$$

This leads to the primary open question regarding VC-minimal theories (and VC-codensity), a restatement of Open Question 1.3 in this terminology.

Open Question 2.11 (VC-codensity in VC-minimal theories)**.** *Is it true that, in all VC-minimal theories T, for all $n < \omega$, $\pi_T(n) = n$?*

In the next subsection, in light of Remark 2.10, we will be working with quasi-forests $\mathcal{F}$, keeping in mind that these quasi-forests will correspond to Δ-virtual types spaces, hence aid us in computing the VC-codensity of formulas in T.

2.3. **Quasi-forests.** Let $(F; \trianglelefteq)$ be a finite quasi-forest. Since F is finite, we can define the *levels* of $(F; \trianglelefteq)$ recursively as follows:

- $\mathrm{Lev}_0(F, \trianglelefteq) = \{a \in F : \text{for all } b \in F, \text{ if } b \trianglelefteq a, \text{then } a \trianglelefteq b\}$.
- $\mathrm{Lev}_n(F, \trianglelefteq) = \mathrm{Lev}_0\left(F \setminus \left(\bigcup_{i<n} \mathrm{Lev}_i(F, \trianglelefteq)\right), \trianglelefteq\right)$ for all $n > 0$.

For each $t \in F$, define $\nu_t : F \to 2$ so that, for all $s \in F$, $\nu_t(s) = 1$ if and only if $s \trianglelefteq t$ and let $\nu_0 : F \to 2$ be the constant zero function. In the model theory context, if we think of t as the parameter, then ν_t is the generic virtual type corresponding to t. As ν_t is a function from F to 2, we will identify it with a subset of F. The "tree of virtual types," is

$$\mathcal{V}(F, \trianglelefteq) := \{\nu_t : t \in F\} \cup \{\nu_0\}$$

ordered via inclusion (i.e., for $p, q \in \mathcal{V}(F, \trianglelefteq)$, $p \trianglelefteq q$ if, for all $t \in F$, if $p(t) = 1$, then $q(t) = 1$). Then, it is easy to see that $\mathcal{V}(F, \trianglelefteq) \setminus \{\nu_0\}$ is isomorphic to the partial order generated by $(F, \trianglelefteq)$ via the map ν. Moreover, for $p, q \in \mathcal{V}(F, \trianglelefteq)$, define $p \wedge q : F \to 2$ so that, for all $t \in F$,

$(p \wedge q)(t) = 1$ if and only if $p(t) = q(t) = 1$. Then, $p \wedge q \in \mathcal{V}(F, \trianglelefteq)$ and $p \wedge q$ is the tree-theoretic meet of p and q.

From an arbitrary linear ordering $\leq^*$ on each level of $\mathcal{V}(F, \trianglelefteq)$, we construct a linear ordering on $\mathcal{V}(F, \trianglelefteq)$, $\leq$, extending the tree order $\trianglelefteq$ as follows:

- If $p \triangleleft q$, then $p < q$.
- If p and q are $\trianglelefteq$-incomparable, let p^* be such that $p \wedge q \triangleleft p^* \trianglelefteq p$ and p^* $\trianglelefteq$-minimal such and similarly define q^*. Then $p < q$ if $p^* <^* q^*$.

Note that in the case where $(F; \trianglelefteq) = \mathcal{F}(C, \Delta)$ for a directed set of formulas $\Delta(x; z)$ and $C \subseteq \mathcal{U}_z$ as above, the ordering $\leq$ we get here corresponds to the "convex ordering" (see [7, 9]). That is, the instances of δ are convex in this ordering. Formally:

Lemma 2.12. *For all $t \in F$, the set $\chi(t) := \{p \in \mathcal{V}(F, \trianglelefteq) : p(t) = 1\}$ is $\leq$-convex.*

Proof. Suppose $t \in F$ and $p < r < q$ with $p, q \in \chi(t)$. In particular, $(p \wedge q)(t) = 1$. If $p \wedge q \trianglelefteq r$, then $r(t) = 1$ so $r \in \chi(t)$. So suppose this fails. If $r \triangleleft p \wedge q$, then in particular $r \triangleleft p$, hence $r < p$, a contradiction. Thus r and $p \wedge q$ are $\trianglelefteq$-incomparable. Since $\mathcal{V}(F, \trianglelefteq)$ is a tree and $r \wedge p, p \wedge q \trianglelefteq p$, we conclude that $r \wedge p \triangleleft p \wedge q$. Thus, $r \wedge p \triangleleft q$, hence $r \wedge p \trianglelefteq r \wedge q$. By symmetry, we obtain $r \wedge p = r \wedge q \triangleleft p \wedge q$. Hence, by the second part of the definition of ordering, either $r < p, q$ or $p, q < r$ (depending on $\leq^*$). Contradiction. $\square$

For each $p, q \in \mathcal{V}(F, \trianglelefteq)$, define

$$\operatorname{diff}(p, q) := \{t \in F : p(t) \neq q(t)\} \text{ and } \operatorname{dist}(p, q) := |\operatorname{diff}(p, q)|.$$

Lemma 2.13. *For any sequence $p_0 < \cdots < p_m$ from $\mathcal{V}(F, \trianglelefteq)$,*

$$\sum_{i<m} \operatorname{dist}(p_i, p_{i+1}) \leq 2|F|.$$

Proof. For any $t \in F$, for all $i < m$, $t \in \operatorname{diff}(p_i, p_{i+1})$ if and only if

- $p_i \in \chi(t)$ and $p_{i+1} \notin \chi(t)$, or
- $p_i \notin \chi(t)$ and $p_{i+1} \in \chi(t)$.

By Lemma 2.12, $\chi(t)$ is $\leq$-convex, so, for each $t \in F$, there exists at most two $i < m$ such that $t \in \operatorname{diff}(p_i, p_{i+1})$. The conclusion follows. $\square$

2.4. **The quasi-forest $\mathcal{F}(C, \Delta)$.** Fix $\Delta(x; z)$ a finite directed set, $C \subseteq \mathcal{U}_z$ finite, and consider $\mathcal{F}(C, \Delta)$ as defined above. Notice that $\mathcal{V}_\Delta(C)$ is isomorphic to $\mathcal{V}(\mathcal{F}(C, \Delta))$. Thus, for $p, q \in \mathcal{V}_\Delta(C)$, we define

$$\operatorname{diff}(p, q) := \{(c, \delta) \in C \times \Delta : p(c, \delta) \neq q(c, \delta)\}$$

and $\operatorname{dist}(p, q) := |\operatorname{diff}(p, q)|$. Clearly this corresponds via our isomorphism to the definition above. By Lemma 2.13, we get the following.

Lemma 2.14. *There exists $\leq$ a linear order on $\mathcal{V}_\Delta(C)$ such that, for all $p_0 < \cdots < p_m$ from $\mathcal{V}_\Delta(C)$,*

$$\sum_{i<m} \operatorname{dist}(p_i, p_{i+1}) \leq 2|C||\Delta|.$$

This lemma is vital to our method of counting types in VC-minimal theories, as we will demonstrate in the next section using the test case of fully VC-minimal theories.

3. Test Case: Fully VC-Minimal Theories

Definition 3.1 (Definition 3.9 of [9]). A theory T is *fully VC-minimal* if there exists a directed family of formulas Δ (with $|x| = 1$) such that, for all formulas $\varphi(x; y)$ with $|x| = 1$ and y arbitrary, $\varphi(x; y)$ is T-equivalent to a boolean combination of elements of Δ.

As noted in Remark 2.9 above, if T is VC-minimal and $\mathrm{acl}^{\mathrm{eq}} = \mathrm{dcl}^{\mathrm{eq}}$, then T is fully VC-minimal. For example, any weakly o-minimal theory is fully VC-minimal. On the other hand, ACVF and even ACF are not fully VC-minimal. See Example 3.15 of [9] for details.

Theorem 3.2 (Theorem 3.14 of [9]). *If T is fully VC-minimal, then $\pi_T(n) = n$ for all $n < \omega$. That is, for all formulas $\varphi(x; y)$, the VC-codensity of φ is $\leq |x|$.*

The proof presented in [9] goes through UDTFS-rank, similar to the proof for weakly o-minimal theories given in [2], but in this section, we will sketch an alternate proof using "pure combinatorics." We use this to motivate the process by which we compute the VC-codensity of some formulas in general VC-minimal theories.

We prove Theorem 3.2 by induction on n. If $n = 1$, fix $\varphi(x; y)$ with $|x| = 1$. Fix a finite directed $\Delta(x; y)$ such that $\varphi(x; y)$ is a boolean combination of elements of Δ. Then, for any finite $B \subseteq \mathcal{U}_y$,

$$|S_\varphi(B)| \leq |S_\Delta(B)|.$$

However, as argued above, $|S_\Delta(B)| \leq |\Delta| \cdot |B| + 1$, which is linear in $|B|$. Hence, $\mathrm{vc}^*(\varphi) \leq 1$.

In general, fix $n > 1$ and consider $\varphi(x_0, x_1; y)$, where $|x_0| = 1$ and $|x_1| = n - 1$. Repartition φ via

$$\hat{\varphi}(x_0; x_1, y) = \varphi(x_0, x_1; y)$$

and, as before, there exists a finite directed $\Delta_0(x_0; x_1, y)$ such that $\hat{\varphi}$ is a boolean combination of elements of Δ_0. Again, for any finite $B \subseteq \mathcal{U}_y$ and any $a_1 \in \mathcal{U}_{x_1}$,

$$|S_{\hat{\varphi}}(a_1{}^\frown B)| \leq |S_{\Delta_0}(a_1{}^\frown B)| \leq |\Delta_0| \cdot |B| + 1.$$

But how do we use this to count φ-types over B instead of $\hat{\varphi}$-types over $a_1{}^\frown B$? We describe the quasi-forest structure given by $\Delta_0(x_0; a_1, B)$.

For each $\delta(x_0; x_1, y), \delta'(x_0; x_1, y) \in \Delta_0$, let

$$\psi_{\delta,\delta'}(x_1; y, y') := \forall x_0 \left(\delta'(x_0; x_1, y') \to \delta(x_0; x_1, y)\right).$$

Notice that, for all $a_1 \in \mathcal{U}_{x_1}$, for all $b, b' \in B$, and for all $\delta, \delta' \in \Delta_0$,

$$\models \psi_{\delta,\delta'}(a_1; b, b') \text{ if and only if } \langle a_1, b, \delta\rangle \trianglelefteq \langle a_1, b', \delta'\rangle,$$

with the quasi-forest structure $\mathcal{F}(a_1{}^\frown B, \Delta_0)$ described in Remark 2.10. Let $\Psi = \{\psi_{\delta,\delta'} : \delta, \delta' \in \Delta_0\}$.

Lemma 3.3 (Quasi-forests determined by Ψ-types)**.** *If $p(x_1) \in S_\Psi(B \times B)$, $a_1, a_1' \models p$, then, as quasi-forests,*

$$\mathcal{F}(a_1{}^\frown B, \Delta_0) \cong \mathcal{F}(a_1'{}^\frown B, \Delta_0)$$

via the map $\langle a_1, b, \delta\rangle \mapsto \langle a_1', b, \delta\rangle$.

Proof. For all $b, b' \in B$, $\delta, \delta' \in \Delta_0$,

$$\langle a_1, b, \delta\rangle \trianglelefteq \langle a_1, b', \delta'\rangle \text{ if and only if } p(x_1) \vdash \psi_{\delta,\delta'}(x_1, b, b').$$

Since the same holds for a_1', we get $\langle a_1, b, \delta\rangle \trianglelefteq \langle a_1, b', \delta'\rangle$ if and only if $\langle a_1', b, \delta\rangle \trianglelefteq \langle a_1', b', \delta'\rangle$. □

In particular, for any partial type p that implies a type in $S_\Psi(B \times B)$, we can define the quasi-forest

$$\mathcal{F}(p, B, \Delta_0) = (B \times \Delta_0; \trianglelefteq_p),$$

where, for all $b, b' \in B$, $\delta, \delta' \in \Delta_0$,

$$\langle b, \delta\rangle \trianglelefteq_p \langle b', \delta'\rangle \text{ if } p(x_1) \vdash \psi_{\delta,\delta'}(x_1; b, b').$$

In particular, for all $a_1 \models p$,

$$\mathcal{F}(p, B, \Delta_0) \cong \mathcal{F}(a_1{}^\frown B, \Delta_0)$$

via the map g_{a_1}, where $g_{a_1}(b, \delta) = \langle a_1, b, \delta\rangle$ for all $b \in B$ and $\delta \in \Delta$. Similar to the definition of $\nu_{c,\delta}$ as in Remark 2.10 above, for $\langle b, \delta\rangle \in \mathcal{F}(p, B, \Delta_0)$, define $\nu_{p,b,\delta} : B \times \Delta_0 \to 2$ so that, given $b' \in B$ and $\delta' \in \Delta_0$,

$$\nu_{p,b,\delta}(b', \delta') := \begin{cases} 1 & \text{if } \langle b', \delta'\rangle \trianglelefteq_p \langle b, \delta\rangle, \\ 0 & \text{otherwise} \end{cases}.$$

That is, $\nu_{p,b,\delta}(b',\delta') = 1$ if and only if

$$p(x_1) \vdash (\forall x_0)(\delta(x_0; x_1, b) \to \delta'(x_0; x_1, b')).$$

To deal with the 0 node, define $\nu_0 : B \times \Delta_0 \to 2$ to be the constant zero function. Moreover, as we did in Remark 2.10, define the *virtual type space*

$$\mathcal{V}_{\Delta_0}(p, B) := \{\nu_{p,b,\delta} : \langle b, \delta\rangle \in \mathcal{F}(p, B, \Delta_0)\} \cup \{\nu_0\}.$$

Then, for $a_1 \models p$,

$$\nu_{b,\delta} \circ g_{a_1} = \nu_{p,b,\delta}.$$

Thus, under precomposition with g_{a_1},

$$\mathcal{V}_{\Delta_0}(a_1 {}^\frown B) \cong \mathcal{V}_{\Delta_0}(p, B).$$

Therefore, we get the following lemma.

Lemma 3.4 (Δ_0-types determined by Ψ-types)**.** *If $p(x_1) \in S_\Psi(B \times B)$ and $a_1 \models p$, then*

$$\{\eta(p) \circ g_{a_1} : p \in S_{\Delta_0}(a_1 {}^\frown B)\} \subseteq \mathcal{V}_{\Delta_0}(p, B).$$

In particular,

$$\eta\left(S_{\Delta_0}(B)\right) \subseteq \bigcup_{p \in S_\Psi(B \times B)} \mathcal{V}_{\Delta_0}(p, B).$$

Hence, without any more work, we get the bound

$$|S_\varphi(B)| \leq |S_{\Delta_0}(B)| \leq (|\Delta_0| \cdot |B| + 1)|S_\Psi(B \times B)|.$$

With no further analysis, induction would yield $\pi_T(n) \leq 2^n - 1$.

Moving forward, we will only consider the case where $n = 2$ and hence $|x_1| = 1$. Now, by full VC-minimality, there exists a finite directed $\Delta_1(x_1; y, y')$ such that each $\psi(x_1; y, y') \in \Psi$ is a boolean combination of elements of Δ_1. Therefore,

$$|S_\varphi(B)| \leq (|\Delta_0| \cdot |B| + 1)|S_{\Delta_1}(B \times B)| \leq$$
$$(|\Delta_0| \cdot |B| + 1) \cdot (|\Delta_1| \cdot |B|^2 + 1) = \mathcal{O}(|B|^3).$$

In other words, $\pi_T(2) \leq 3$. We can get $\pi_T(2) = 2$ by paying closer attention to our counting.

Apply Lemma 2.14 to $B \times B$ and Δ_1. Let $p_0 < \cdots < p_m$ enumerate $S_{\Delta_1}(B \times B)$ inside $\mathcal{V}_{\Delta_1}(B \times B)$ by identifying p_i and $\eta(p_i)$; hence,

$$\sum_{i<m} \operatorname{dist}(p_i, p_{i+1}) \leq 2|B|^2|\Delta_1|.$$

For each $\langle b, b', \delta_1\rangle \in \operatorname{diff}(p_i, p_{i+1})$, formulas of the form $\psi_{\delta_0,\delta_0'}(x_1; b, b')$ for $\delta_0, \delta_0' \in \Delta_0$ are potentially changed between p_i and p_{i+1}. That is, either

- $\langle b, \delta_0 \rangle \trianglelefteq_{p_i} \langle b', \delta'_0 \rangle$ and $\langle b, \delta_0 \rangle \ntrianglelefteq_{p_{i+1}} \langle b', \delta'_0 \rangle$, or
- $\langle b, \delta_0 \rangle \ntrianglelefteq_{p_i} \langle b', \delta'_0 \rangle$ and $\langle b, \delta_0 \rangle \trianglelefteq_{p_{i+1}} \langle b', \delta'_0 \rangle$.

Thus, for each $\langle b, b', \delta_1 \rangle \in \operatorname{diff}(p_i, p_{i+1})$, we get at most $|\Delta_0|$ new virtual Δ_0-types in the corresponding virtual Δ_0-type space $\mathcal{V}_{\Delta_0}(p_{i+1}, B)$, namely $\nu_{p_{i+1},b',\delta'_0}$ for all $\delta'_0 \in \Delta_0$ (i.e., from the change in $\nu_{p_{i+1},b',\delta'_0}(b, \delta_0)$ versus $\nu_{p_i,b',\delta'_0}(b, \delta_0)$ for each $\delta_0 \in \Delta_0$). Therefore,

$$|\mathcal{V}_{\Delta_0}(p_{i+1}, B) \setminus \mathcal{V}_{\Delta_0}(p_i, B)| \leq |\Delta_0||\operatorname{diff}(p_i, p_{i+1})| = |\Delta_0|\operatorname{dist}(p_i, p_{i+1}).$$

Therefore,

$$\begin{aligned}
&\left| \bigcup_{p \in S_{\Delta_1}(B \times B)} \mathcal{V}_{\Delta_0}(p, B) \right| \leq \\
&|\mathcal{V}_{\Delta_0}(p_0, B)| + \sum_{i<m} |\mathcal{V}_{\Delta_0}(p_{i+1}, B) \setminus \mathcal{V}_{\Delta_0}(p_i, B)| \leq \\
&|\mathcal{V}_{\Delta_0}(p_0, B)| + |\Delta_0| \sum_{i<m} \operatorname{dist}(p_i, p_{i+1}) \leq \\
&|\mathcal{V}_{\Delta_0}(p_0, B)| + 2|B|^2|\Delta_1||\Delta_0| \leq \\
&2|B|^2|\Delta_1||\Delta_0| + |B||\Delta_0| + 1.
\end{aligned}$$

In particular,

$$|S_\varphi(B)| = \mathcal{O}(|B|^2).$$

Therefore, $\pi_T(2) = 2$. The argument is similar for $n > 2$.

4. General VC-Minimal Theories

In the general case, by Theorem 2.7, we can assume that the formula whose VC-codensity we are computing is such that each instance is T-equivalent to a union of a uniformly bounded number of balls. However, the problem comes in distinguishing these balls from one another since, in general, they are not individually definable over the parameter used in the instance considered. So we will need some way of determining irreducible unions of balls and do this relative to a given type-space.

For the remainder of this section, we will give a proof of Theorem 1.4, following the outline sketched in Section 3. That is, if T is a VC-minimal theory, then we will show that

$$\pi_T(2) = 2.$$

4.1. **Construction Setup.** Fix T a VC-minimal theory and $\varphi(x; y)$ is a partitioned formula with $|x| = 2$. Repartition as

$$\hat{\varphi}(x_0; x_1, y) := \varphi(x_0, x_1; y).$$

By Theorem 2.7, there exists $N_0 < \omega$, $\delta_0(x_0; z)$ directed, and $\Gamma_0(x_0; x_1, y)$ a finite set of formulas such that

- $\hat{\varphi}(x_0; x_1, y)$ is T-equivalent to a boolean combination of elements of Γ_0, and
- each instance of a formula from Γ_0 is T-equivalent to a union of at most N_0 instances of δ_0.

Fix a finite $B \subseteq \mathcal{U}_y$ and we aim to count the size of $S_\varphi(B)$. As each type in $S_\varphi(B)$ is implied by a type in $S_{\Gamma_0}(B)$, we have

$$|S_\varphi(B)| \leq |S_{\Gamma_0}(B)|,$$

so we will count Γ_0-types over B instead (correctly repartitioned). Each Γ_0-type is, in fact, determined by an instance of δ_0, just not necessarily definably over B. For each $a_1 \in \mathcal{U}_{x_1}$ and each $d \in \mathcal{U}_z$, define the generic virtual Γ_0-type of $\delta_0(x_0; d)$ over $a_1 {}^\frown B$, $\nu_{a_1,d} : B \times \Gamma_0 \to 2$, by setting, for each $b \in B$ and $\gamma \in \Gamma_0$,

$$\nu_{a_1,d}(b, \gamma) = \begin{cases} 1 & \text{if } \models \forall x_0(\delta_0(x_0; d) \to \gamma(x_0; a_1, b)), \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\mathcal{V}(a_1, B) := \{\nu_{a_1,d} : d \in \mathcal{U}_z\} \cup \{\nu_0\},$$

where, as before, $\nu_0 : B \times \Gamma_0 \to 2$ is the constant zero function. Then, it is easy to check that

$$\eta(S_{\Gamma_0}(B)) \subseteq \bigcup \{\mathcal{V}(a_1, B) : a_1 \in \mathcal{U}_{x_1}\}. \tag{1}$$

Therefore, it suffices to bound this set. As we did in Section 3, we will use types in the x_1 variable to bound this. We code this now.

For each formula $\gamma \in \Gamma_0$, each $a_1 \in \mathcal{U}_{x_1}$, and each $b \in B$, $\gamma(x_0; a_1, b)$ is T-equivalent to a union of at most N_0 instances of δ_0. If $\gamma(x_0, a_1, b)$ is T-equivalent to $\bigvee_{i<n} \delta_0(x_0; d_i)$ for $d_i \in \mathcal{U}_z$ with $n \leq N_0$ minimal such, then we will call the $\delta_0(x_0; d_i)$'s *components* of $\gamma(x_0; a_1, b)$. Note that, by directedness and minimality of n, components are unique up to permutation and T-equivalence.

With this in mind, for each $n \leq N_0$ and each $\gamma \in \Gamma_0$, let

$$\psi''_{\gamma,n}(x_1, y, z_0, \ldots, z_{n-1}) := \forall x_0 \left(\gamma(x_0; x_1, y) \leftrightarrow \bigvee_{i<n} \delta_0(x_0; z_i) \right)$$

and let

$$\psi'_{\gamma,n}(x_1, y) := (\exists z_i)_{i<n}[\psi''_{\gamma,n}] \wedge \neg(\exists z_i)_{i<n-1}[\psi''_{\gamma,n-1}].$$

Then, $\models \psi'_{\gamma,n}(a_1, b)$ if and only if $\gamma(x_0; a_1, b)$ has exactly n components.

The next step is to code the virtual Γ_0-types that correspond to the generic type of some component of $\gamma(x_0; a_1, b)$. For each $m < \omega$, $n \leq N_0$, and $\mu \subseteq {}^{m\times\Gamma_0}2$, let

$$\begin{aligned}&\psi_{\gamma,n,m,\mu}(x_1, y, w_0, \ldots, w_{m-1}) := \psi'_{\gamma,n}(x_1, y)\wedge\\&(\exists z_i)_{i<n}\Big[\psi''_{\gamma,n}(x_1, y, z_0, \ldots, z_{n-1})\wedge\\&\bigwedge_{s\in {}^{m\times\Gamma_0}2}\left(\bigvee_{i<n}\bigwedge_{j<m,\gamma'\in\Gamma_0}[\forall x_0(\delta_0(x_0; z_i) \to \gamma'(x_0; x_1, w_j))]^{s(j,\gamma')}\right)^{s\in\mu}\Big].\end{aligned}$$

We demonstrate how $\psi_{\gamma,n,m,\mu}$ codes the desired virtual Γ_0-type space. Fix $a_1 \in \mathcal{U}_{x_1}$, $b \in B$, $c_0, \ldots, c_{m-1} \in B$ and, any $d_0, \ldots, d_{n-1} \in \mathcal{U}_z$ such that $\{\delta_0(x_0; d_i) : i < n\}$ is the set of components of $\gamma(x_0; a_1, b)$. Let $g_{\bar{c}} : m \times \Gamma_0 \to B \times \Gamma_0$ be given by $g_{\bar{c}}(j, \gamma') = \langle c_j, \gamma'\rangle$ for all $j < m$ and $\gamma' \in \Gamma_0$. Let

$$\mu = \{\nu_{a_1,d_i} \circ g_{\bar{c}} : i < n\}.$$

So μ codes the set, over all $i < n$, of the generic virtual Γ_0-type of $\delta_0(x_0; d_i)$ over $a_1{}^\frown\{c_j : j < m\}$ (via $g_{\bar{c}}$). Clearly $|\mu| \leq n$ (and it could be strictly less than n if more than one component codes the same virtual type). Therefore, we get

$$\models \psi_{\gamma,n,m,\mu}(a_1, b, c_0, \ldots, c_{m-1}).$$

Moreover, both $n \leq N_0$ and $\mu \subseteq {}^{m\times\Gamma_0}2$ are unique such for the given $a_1, b, c_0, \ldots, c_{m-1}$. For each $m < \omega$, define

$$\Psi_m(x_1; y, w_0, \ldots, w_{m-1}) := \{\psi_{\gamma,n,m,\mu} : \gamma \in \Gamma_0, n \leq N_0, \mu \subseteq {}^{m\times\Gamma_0}2\}$$

and let $\Psi := \Psi_{2^{N_0}}$ (our choice to consider only $m \leq 2^{N_0}$ will be made clear shortly). By VC-minimality, there exists $\delta_1(x_1; u)$ directed and $N_1 < \omega$ such that every instance of Ψ is boolean combination of at most N_1 instances of δ_1. Note that this also covers instances of Ψ_m for $m < 2^{N_0}$ by repeating entries.

The goal now is to build a Ψ-type space over a set of size $\mathcal{O}(|B|^2)$ such that each type determines a virtual Γ_0-type space, $\mathcal{V}(a_1, B)$. Then, as we did in Section 3, we use this to bound the size of Γ_0-types over B. To do this, we will need some definitions about how to relate various Γ_0-type spaces via Ψ.

Definition 4.1. Fix $b \in B$, $\gamma \in \Gamma_0$, $m < \omega$, $\overline{c} \in B^m$, and $p(x_1)$ any partial type. We say that p *decides generic* Γ_0*-types of* $\gamma(x_0; x_1, b)$ *over* $\overline{c}$ if, for some $n < \omega$ and $\mu \subseteq {}^{m \times \Gamma_0}2$,

$$p(x_1) \vdash \psi_{\gamma,n,m,\mu}(x_1, b, \overline{c}).$$

In this case, let $\mu_{p,\gamma,b,\overline{c}}$ be the unique such μ, which will code the type space. Also, let $N_{p,\gamma,b,\overline{c}} := |\mu|$, which denotes the size of the type space.

If $p(x_1)$ decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\overline{c}$, $a_1 \models p_1$, and $\{\delta_0(x_0; d_i) : i < n\}$ is the set of components of $\gamma(x_0; a_1, b)$, then $\mu_{p,\gamma,b,\overline{c}} = \{\nu_{a_1,d_i} \circ g_{\overline{c}} : i < n\}$. Clearly if $p(x_1)$ implies a type in $S_{\Psi_m}(\{\langle b, \overline{c}\rangle\})$, then p decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\overline{c}$ for any $\gamma \in \Gamma_0$. Moreover, if p decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\overline{c}$ and $\overline{c}_0 \subseteq \overline{c}$ is any subsequence, then p decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\overline{c}_0$ as well.

Note that $N_{p,\gamma,b,\overline{c}} \leq N_0$ for any choice of b, γ, $\overline{c}$, and p that decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\overline{c}$. This is because, for the formula $\psi_{\gamma,n,|\overline{c}|,\mu}(x_1; b, \overline{c})$ to even be consistent, we must have $N_{p,\gamma,b,\overline{c}} = |\mu| \leq n \leq N_0$.

Fix b, γ, $\overline{c}$, and p such that p decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\overline{c}$ and fix $\overline{c}_0 \subseteq \overline{c}$. Let $\pi_{p,\gamma,b,\overline{c},\overline{c}_0} : \mu_{p,\gamma,b,\overline{c}} \to \mu_{p,\gamma,b,\overline{c}_0}$ be the projection map. That is, if $\overline{c} = \langle c_i : i < m\rangle$ and $\overline{c}_0 = \langle c_{i_0}, \dots, c_{i_{k-1}}\rangle$ for $i_0 < \cdots < i_{k-1} < n$ and $s \in \mu_{p,\gamma,b,\overline{c}}$, then

$$\pi_{p,\gamma,b,\overline{c},\overline{c}_0}(s)(j, \gamma') = s(i_j, \gamma').$$

In general, this map is surjective and, when $N_{p,\gamma,b,\overline{c}} = N_{p,\gamma,b,\overline{c}_0}$, it is bijective. This leads us to the following definition.

Definition 4.2. Fix $b \in B$, $\gamma \in \Gamma_0$, $m, m' < \omega$, $\overline{c} \in B^m$, and $\overline{c}' \in B^{m'}$, and $p(x_1)$ any partial type. If p decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\overline{c}^\frown\overline{c}'$, then we say that $\overline{c}$ and $\overline{c}'$ *generate the same irreducibles of* $\gamma(x_0; x_1, b)$ *with respect to* p if

$$N_{p,\gamma,b,\overline{c}} = N_{p,\gamma,b,\overline{c}^\frown\overline{c}'} = N_{p,\gamma,b,\overline{c}'}.$$

If $\overline{c}$ and $\overline{c}'$ generate the same irreducibles of $\gamma(x_0; x_1, b)$ with respect to p, then the map

$$\rho_{p,\gamma,b,\overline{c},\overline{c}'} := \pi^{-1}_{p,\gamma,b,\overline{c}^\frown\overline{c}',\overline{c}} \circ \pi_{p,\gamma,b,\overline{c}^\frown\overline{c}',\overline{c}'}$$

is a canonical bijection between $\mu_{p,\gamma,b,\overline{c}}$ and $\mu_{p,\gamma,b,\overline{c}'}$. Thus, we can use the information of Ψ-types over small parts of B and "glue" this information together via these bijections to get information about the Γ_0-type over all of B. We detail this construction now.

4.2. **Primary Construction.** We are now ready to begin the primary construction. Fix $b \in B$, $\gamma \in \Gamma_0$, and put $<$ an arbitrary linear order on B. For each $k \leq N_0$, will recursively define $\beta_k \subseteq B^{2^k}$ (depending on b and γ). Let

$$S_k := S_{\Psi_{2^k}}(b^\frown \beta_k).$$

If $k \geq 1$, we maintain that

if $q \in S_k$, then there exists $p \in S_{k-1}$ such that $q \vdash p$.

We also construct, for all $p \in S_{k-1}$, $\beta_{p,k} \subseteq \beta_k$, which is linearly ordered by the first element of the sequence. Finally, we construct, for all $q \in S_k$ and $p \in S_{k-1}$ with $q \vdash p$, $\beta^q_{p,k} \subseteq \beta_{p,k}$.

First, let $\beta_0 := B$, which is ordered by our arbitrary $<$. Note that, for each $p \in S_0$, p decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over b' for all $b' \in B$. Therefore, for each $p \in S_0$, the set

$$\beta^p_0 := \{b' \in \beta_0 : N_{p,\gamma,b,b'} > 1\}$$

is well-defined. This corresponds to the elements b' such that, according to p, there is more than one Γ_0-type over b' implied by some component of $\gamma(x_0; x_1, b)$. Let

$$\beta_{p,1} := \{\langle b', b''\rangle : b', b'' \in \beta^p_0, b' < b'' \text{ are } <\text{-consecutive}\},$$

which inherits the first element linear order $<$ from β^p_0. Let

$$\beta_1 := \left(\bigcup\{\beta_{p,1} : p \in S_0\}\right) \cup \{\langle b', b'\rangle : b' \in \beta_0\}.$$

By including a copy of the diagonal of β_0 in β_1, we ensure that, for each $q \in S_1$, there exists $p \in S_0$ such that $q \vdash p$, as desired.

Fix $k \geq 1$ and assume we have carried out our construction up to stage k. Fix $q \in S_k$ and $p \in S_{k-1}$ such that $q \vdash p$. Define

$$\beta^q_{p,k} := \{\overline{c} \in \beta_{p,k} : N_{q,\gamma,b,\overline{c}} > N_{q,\gamma,b,\overline{c}_0} \text{ or } N_{q,\gamma,b,\overline{c}} > N_{q,\gamma,b,\overline{c}_1}\},$$

where $\overline{c}_0$ is the first half of $\overline{c}$ and $\overline{c}_1$ is the second half of $\overline{c}$. That is, $\overline{c}_0$ and $\overline{c}_1$ do not generate the same irreducibles of $\gamma(x_0; x_1, b)$ with respect to q. Define

$$\beta_{q,k+1} := \{\overline{c}^\frown \overline{c}' : \overline{c}, \overline{c}' \in \beta^q_{p,k}, \overline{c} < \overline{c}' \text{ are } <\text{-consecutive}\},$$

which inherits the first element linear order $<$ from $\beta^q_{p,k}$. Let

$$\beta_{k+1} := \left(\bigcup\{\beta_{q,k+1} : q \in S_k\}\right) \cup \{\overline{c}^\frown \overline{c} : \overline{c} \in \beta_k\}.$$

Notice that, as we included a copy of the diagonal of β_k in β_{k+1}, for each $q \in S_{k+1}$, there exists $p \in S_k$ such that $q \vdash p$.

This all depends on $b \in B$ and $\gamma \in \Gamma_0$, so define $\beta_{\gamma,b} := \beta_{N_0}$ (Lemma 4.3 below explains why we choose to stop at $k = N_0$). Let

$$\beta := \bigcup \{b^\frown \beta_{\gamma,b} : \gamma \in \Gamma_0, b \in B\},$$

and let $S := S_\Psi(\beta)$.

This concludes our construction. We need only show that this works.

4.3. **Verifying Construction Works.** First, its clear that S is the set of Ψ-types over $\beta \subseteq B^{2^{N_0}+1}$. We thus need to check that $|\beta| = \mathcal{O}(|B|^2)$ and, for each $p(x_1) \in S$, for all $a_1, a_1' \models p$, $\mathcal{V}(a_1, B) = \mathcal{V}(a_1', B)$. Moreover, we then need to check that this implies that $|S_\varphi(B)| = \mathcal{O}(|B|^2)$, as desired.

For the next lemma, fix $b \in B$ and $\gamma \in \Gamma_0$.

Lemma 4.3. *The construction terminates by stage $n = N_0$. That is, for all $q \in S_{N_0}$, $\beta^q_{p,N_0} = \emptyset$ (where $p \in S_{N_0-1}$ is such that $q \vdash p$).*

Proof. If any q decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\overline{c}$, then $N_{q,\gamma,b,\overline{c}} \leq N_0$. Since each iteration of the construction increases the value of this $N_{q,\gamma,b,\overline{c}}$ by at least one, it cannot continue past N_0 steps. □

Lemma 4.4. *For all $p(x_1) \in S$, for all $a_1, a_1' \models p$, $\mathcal{V}(a_1, B) = \mathcal{V}(a_1', B)$.*

Proof. Fix $a_1, a_1' \models p$. Fix $b \in B$ and $\gamma \in \Gamma_0$. Fix $n \leq N_0$, $d_0, \ldots, d_{n-1} \in \mathcal{U}_z$, and $d_0', \ldots, d_{n-1}' \in \mathcal{U}_z$ such that $\{\delta_0(x_0; d_i) : i < n\}$ is the set of components of $\gamma(x_0; a_1, b)$ and $\{\delta_0(x_0; d_i') : i < n\}$ is the set of components of $\gamma(x_0; a_1', b)$. Fix $p_0, \ldots, p_{N_0} = p$ such that $p_i \in S_i$ and $p_{i+1} \vdash p_i$ for all $i < N_0$ (as in the primary construction for our choice of b and γ). For simplicity of notation, let $\beta_i^* = \beta^{p_i}_{p_{i-1},i}$ for $1 \leq i \leq N_0$ and let $\beta_0^* = \beta_0^{p_0}$. Choose k such that $\beta_k^* = \emptyset$ and $\beta_{k-1}^* \neq \emptyset$. Such a $k \leq N_0$ exists by Lemma 4.3. Choose $\overline{c}^* \in \beta_{k-1}^*$ to be $<$-minimal. Clearly p decides generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\overline{c}^*$, so let $\mu^* = \mu_{p,\gamma,b,\overline{c}^*}$. Note that, since $a_1, a_1' \models p$,

$$\mu^* = \{\nu_{a_1,d_i} \circ g_{\overline{c}^*} : i < n\} = \left\{\nu_{a_1',d_i'} \circ g_{\overline{c}^*} : i < n\right\}.$$

We will now show that, in fact,

$$\{\nu_{a_1,d_i} : i < n\} = \left\{\nu_{a_1',d_i'} : i < n\right\}.$$

Fix $s \in \mu^*$ and define $s_* : B \times \Gamma_0 \to 2$ as follows: Fix $b' \in B$ and $\gamma' \in \Gamma_0$. If $b' \in \overline{c}^*$, then set $s_*(b', \gamma') = s(i, \gamma')$, where b' is the ith element of $\overline{c}^*$. If $b' \notin \beta_0^*$, then $N_{p,\gamma,b,b'} = 1$ by definition, hence there is a unique $s' \in \mu_{p,\gamma,b,b'}$. Let $s_*(b', \gamma') = s'(0, \gamma')$. Otherwise, choose $0 < m < k$ maximal such that $b' \in \overline{c} \in \beta_m^*$. We define a surjective function $h_{\overline{c}} : \mu^* \to \mu_{p,\gamma,b,\overline{c}}$ by reverse induction on m as follows:

We do the base case (when $m = k-1$) and the inductive step simultaneously. First, choose $\overline{c}' \in \beta^*_m$ such that

- if $m = k-1$, $\overline{c}' = \overline{c}^*$, and
- if $m < k-1$, $\overline{c}' \subseteq \overline{c}'' \in \beta^*_{m+1}$ and $\overline{c}'$ is $<$-closest such to $\overline{c}$.

So there exists a chain of $<$-consecutive elements in β^*_m

$$\overline{c}_0 < \cdots < \overline{c}_\ell$$

such that $\overline{c}_0 = \overline{c}'$ and $\overline{c}_\ell = \overline{c}$ or vice versa. By choice of $\overline{c}'$, for each $j < \ell$, $\overline{c}_j {}^\frown \overline{c}_{j+1} \in \beta_{p_m, m+1}$ yet $\overline{c}_j {}^\frown \overline{c}_{j+1} \notin \beta^*_{m+1}$. Thereby, we have $N_{p,\gamma,b,\overline{c}_j} = N_{p,\gamma,b,\overline{c}_j {}^\frown \overline{c}_{j+1}} = N_{p,\gamma,b,\overline{c}_{j+1}}$. In other words, $\overline{c}_j$ and $\overline{c}_{j+1}$ generate the same irreducibles of $\gamma(x_0; x_1, b)$ with respect to p. Hence,

$$\rho_{p,\gamma,b,\overline{c}_0,\overline{c}_1} \circ \cdots \circ \rho_{p,\gamma,b,\overline{c}_{\ell-1},\overline{c}_\ell}$$

is a bijection between $\mu_{p,\gamma,b,\overline{c}'}$ and $\mu_{p,\gamma,b,\overline{c}}$ (or vice versa). If $m = k-1$, set $h_{\overline{c}}$ to be this bijection from $\mu^* = \mu_{p,\gamma,b,\overline{c}'}$ to $\mu_{p,\gamma,b,\overline{c}}$. If $m < k-1$, then there exists a surjection $\mu_{p,\gamma,b,\overline{c}''}$ onto $\mu_{p,\gamma,b,\overline{c}}$ by composing the bijection with $\pi_{p,\gamma,b,\overline{c}'',\overline{c}'}$. Compose this with $h_{\overline{c}''}$ from the induction hypothesis to obtain the surjection $h_{\overline{c}} : \mu^* \to \mu_{p,\gamma,b,\overline{c}}$.

Once $h_{\overline{c}}$ is defined, set $s_*(b', \gamma') = h_{\overline{c}}(s)(i, \gamma')$, where b' is the ith element of $\overline{c}$. This concludes our construction of s_*.

Unraveling the construction, we see that, for all $i < n$, if $s = \nu_{a_1,d_i} \circ g_{\overline{c}^*}$, then $s_* = \nu_{a_1,d_i}$. Moreover, for all $i < n$, if $s = \nu_{a'_1,d'_i} \circ g_{\overline{c}^*}$, then $s_* = \nu_{a'_1,d'_i}$. Therefore,

$$\{\nu_{a_1,d_i} : i < n\} = \{\nu_{a'_1,d'_i} : i < n\} = \{s_* : s \in \mu^*\}.$$

Since this holds for all $b \in B$ and $\gamma \in \Gamma_0$, we conclude that

$$\mathcal{V}(a_1, B) = \mathcal{V}(a'_1, B).$$

□

In light of this, for any $p \in S$, define $\mathcal{V}(p, B) = \mathcal{V}(a_1, B)$ for any (equivalently all) $a_1 \models p$. From the proof of Lemma 4.4,

$$|\mathcal{V}(p, B)| \leq N_0 \cdot |B| \cdot |\Gamma_0| + 1.$$

From this and (1) we obtain

$$|S_\varphi(B)| \leq |S_{\Gamma_0}(B)| \leq (N_0 \cdot |B| \cdot |\Gamma_0| + 1) \cdot |S|.$$

By VC-minimality, we know that $|S| \leq N_1 |\beta| + 1$, thus we obtain

$$|S_\varphi(B)| = \mathcal{O}(|B| \cdot |\beta|).$$

A priori, there is no good bound on the size of β, so this is not immediately helpful. However, using Lemma 2.14, we will obtain a bound for $|\beta|$ on the order of $|B|^2$.

For the next two lemmas, fix $b \in B$ and $\gamma \in \Gamma_0$.

For each $i \le N_0$, for each $\psi \in \Psi_{2^i}$ and $\bar{c} \in \beta_i$, there exists $D^i_{\psi,\bar{c}} \subseteq \mathcal{U}_u$ with $|D^i_{\psi,\bar{c}}| \le N_1$ so that $\psi(x_1; b, \bar{c})$ is T-equivalent to a boolean combination of $\delta_1(x_1; d)$ for $d \in D^i_{\psi,\bar{c}}$. For each $k \le N_0$, let

$$D_k := \bigcup \{D^i_{\psi,\bar{c}} : i \le k, \psi \in \Psi_{2^i}, \bar{c} \in \beta_i\}.$$

Hence, for any $q \in S_k$, there exists $p \in S_{\delta_1}(D_k)$ such that $p(x_1) \vdash q(x_1)$. For each $k < \omega$, define $M_k < \omega$ recursively as follows: $M_0 := 0$ and $M_{k+1} := 3M_k + 3$ for all $k \ge 0$.

Lemma 4.5. *For all $k \le N_0$, for all $q_0, q_1 \in S_k$ and $p_0, p_1 \in S_{\delta_1}(D_k)$ with $p_0 \vdash q_0$ and $p_1 \vdash q_1$, we have*

$$|\beta_{q_1,k+1} \triangle \beta_{q_0,k+1}| \le M_{k+1} \cdot \mathrm{dist}(p_0, p_1).$$

Proof. By induction on k. In the base case, we note that $|\beta_0 \triangle \beta_0| = |B \triangle B| = 0$. Fix $k \ge 0$ and $t < 2$. If $k = 0$, let $\beta'_t = \beta_0 = B$ and let $\beta''_t = \beta_0^{q_t}$. If $k > 0$, fix $q'_t \in S_{k-1}$ such that $q'_t \vdash q_t$, let $\beta'_t = \beta_{q'_t,k}$ and let $\beta''_t = \beta^{q_t}_{q'_t,k}$. In either case, let $\beta'''_t = \beta_{q_t,k+1}$. Thus, $\beta''_t \subseteq \beta'_t$ and

$$\beta'''_t = \{\bar{c}_0 {}^\frown \bar{c}_1 : \bar{c}_0, \bar{c}_1 \in \beta''_t \text{ are } <\text{-consecutive}\}.$$

We aim to prove $|\beta'''_0 \triangle \beta'''_1| \le M_{k+1} \cdot \mathrm{dist}(p_0, p_1)$ using the induction hypothesis $|\beta'_0 \triangle \beta'_1| \le M_k \cdot \mathrm{dist}(p_0|_{D_{k-1}}, p_1|_{D_{k-1}})$.

We accomplish this goal by creating a path from $\langle \beta'_0, \beta''_0 \rangle$ to $\langle \beta'_1, \beta''_1 \rangle$ and show that each step along the path is "induced" by either an element of $\beta'_0 \triangle \beta'_1$ or by some element of $\mathrm{diff}(p_0, p_1)$ and each step along the path changes only a small number of elements between β'''_0 and β'''_1.

Let $\xi'_0 = \beta'_0$ and $\xi''_0 = \beta''_0$. For the first stage of the construction, at step i, choose $\bar{c} \in \xi'_i \triangle \beta'_1$. If $\bar{c} \in \xi'_i \setminus \beta'_1$, let $\xi'_{i+1} = \xi'_i \setminus \{\bar{c}\}$ and let $\xi''_{i+1} = \xi''_i$. If $\bar{c} \in \beta'_1 \setminus \xi'_i$, let $\xi'_{i+1} = \xi'_i \cup \{\bar{c}\}$ and let $\xi''_{i+1} = \xi''_i \cup (\beta''_1 \cap \{\bar{c}\})$. After $|\beta'_0 \triangle \beta'_1|$ steps, we obtain $\xi'_i = \beta'_1$. This concludes the first stage. For the second stage of the construction, at step i, choose $\bar{c} \in \xi''_i \triangle \beta''_1$. If $\bar{c} \in \xi''_i \setminus \beta''_1$, let $\xi''_{i+1} = \xi''_i \setminus \{\bar{c}\}$. If $\bar{c} \in \beta''_1 \setminus \xi''_i$, let $\xi''_{i+1} = \xi''_i \cup \{\bar{c}\}$. In any case, let $\xi'_{i+1} = \xi'_i$. After a finite number of steps (say $\ell < \omega$), we obtain $\xi''_\ell = \beta''_1$, and the construction terminates. For each $i \le \ell$, let

$$\xi'''_i = \{\bar{c}_0 {}^\frown \bar{c}_1 : \bar{c}_0, \bar{c}_1 \in \xi''_i \text{ are } <\text{-consecutive}\}.$$

Then $\xi'''_0 = \beta'''_0$ and $\xi'''_\ell = \beta'''_1$.

First, we show that, for each $i < \ell$, $|\xi'''_i \triangle \xi'''_{i+1}| \le 3$. This implies

$$|\beta'''_0 \triangle \beta'''_1| \le 3\ell.$$

If $\bar{c} \in \xi''_i \triangle \xi''_{i+1}$, then, if there exists $\bar{c}_0, \bar{c}_1 \in \xi''_i$ (hence in ξ''_{i+1} as well) with $\bar{c}_0 < \bar{c}$ and maximal such and $\bar{c} < \bar{c}_1$ and minimal such, then

$$\xi'''_i \triangle \xi'''_{i+1} = \{\bar{c}_0 {}^\frown \bar{c}, \bar{c} {}^\frown \bar{c}_1, \bar{c}_0 {}^\frown \bar{c}_1\}.$$

If one or both of these do not exist, then clearly $|\xi_i'''\triangle\xi_{i+1}'''| \leq 1$. On the other hand, if $\overline{c} \in \xi_i'\triangle\xi_{i+1}'$ but $\overline{c} \notin \xi_i''\triangle\xi_{i+1}''$, then $\xi_i''' = \xi_{i+1}'''$. By construction, this includes all possible cases.

Next, we show that $\ell \leq |\beta_0'\triangle\beta_1'| + \mathrm{dist}(p_0, p_1)$. In other words, we show that there are at most $\mathrm{dist}(p_0, p_1)$ steps in the second stage of the construction. For each step i in the second stage of the construction, we obtain a unique $\overline{c} \in \xi_i''\triangle\xi_{i+1}''$. However, by construction, we must have that $\overline{c} \in \beta_0' \cap \beta_1'$ and $\overline{c} \in \beta_0''\triangle\beta_1''$ (as $\beta_1' = \xi_{i'}'$ and $(\beta_1' \setminus \beta_0') \cap \xi_{i'}'' = (\beta_1' \setminus \beta_0') \cap \beta_1''$ for all i' in the second stage of the construction). That is, the first half and second half of $\overline{c}$ do not generate the same irreducibles of $\gamma(x_0; x_1, b)$ with respect to one of q_0 or q_1 but do for the other. This implies that there exists $\psi \in \Psi_{2^k}$ such that $\psi(x_1; \overline{c}) \in \mathrm{diff}(q_0, q_1)$. Thus, there exists $d \in D^k_{\psi,\overline{c}}$ such that $\langle d, \delta_1\rangle \in \mathrm{diff}(p_0, p_1)$. Therefore, the second stage has at most $\mathrm{dist}(p_0, p_1)$ steps.

Finally, we obtain

$$|\beta_0'''\triangle\beta_1'''| \leq 3(|\beta_0'\triangle\beta_1'| + \mathrm{diff}(p_0, p_1)) \leq$$
$$3(M_k \cdot \mathrm{diff}(p_0|_{D_{k-1}}, p_1|_{D_{k-1}}) + \mathrm{diff}(p_0, p_1)) \leq M_{k+1} \cdot \mathrm{diff}(p_0, p_1).$$

□

For each $k < \omega$, define $M_k' < \omega$ recursively as follows: Let $M_k' := 1$ and let

$$M_{k+1}' := 2M_k'(1 + (k+1)M_{k+1}N_1|\Psi|)$$

for all $k \geq 0$. Note that this is independent of B. We can employ Lemma 4.5 and Lemma 2.14 to bound the size of β_n uniformly in $|B|$.

Lemma 4.6. *For all $k \leq N_0$, $|\beta_k| \leq M_k' \cdot |B|$.*

Proof. We prove this by induction on k. For $k = 0$, $\beta_0 = B$, hence $|\beta_0| = |B|$, as desired.

Suppose $|\beta_k| \leq M_k' \cdot |B|$ and construct D_k as above. In particular,

$$|D_k| \leq \sum_{i=0}^{k} N_1 \cdot |\Psi_{2^i}||\beta_i| \leq$$
$$(k+1) \cdot N_1 \cdot |\Psi| \cdot |\beta_k| \leq (k+1) \cdot N_1 \cdot |\Psi| \cdot M_k' \cdot |B|.$$

Moreover, for each type $q \in S_k$, there exists $p \in S_{\delta_1}(D_k)$ such that $p \vdash q$. Let $q_0, q_1, \ldots, q_{m-1} \in S_k$ be an enumeration of S_k and, for each $i < m$, choose $p_i \in S_{\delta_1}(D_k)$ such that $p_i \vdash q_i$. By Lemma 4.5,

$$|\beta_{q_{i+1},k+1} \setminus \beta_{q_i,k+1}| \leq |\beta_{q_{i+1},k+1}\triangle\beta_{q_i,k+1}| \leq M_{k+1} \cdot \mathrm{dist}(p_i, p_{i+1}).$$

Moreover, by definition of β_{k+1}, we get

$$\beta_{k+1} = \beta_{q_0,k+1} \cup \bigcup_{i<m-1} (\beta_{q_{i+1},k+1} \setminus \beta_{q_i,k+1}) \cup \{\overline{c}^\frown\overline{c} : \overline{c} \in \beta_k\}.$$

By Lemma 2.14 on $\langle p_i : i < m\rangle$ (possibly reordering so that these form a consecutive sequence), we get

$$|\beta_{k+1}| \leq |\beta_{q_0,k+1}| + M_{k+1} \cdot 2 \cdot |D_k| + |\beta_k|.$$

Now $|\beta_{q_0,k+1}| \leq |\beta_k| \leq M'_k \cdot |B|$ by the induction hypothesis, so

$$|\beta_{k+1}| \leq 2 \cdot M'_k \cdot |B| + M_{k+1} \cdot 2 \cdot (k+1) \cdot N_1 \cdot |\Psi| \cdot M'_k \cdot |B|$$

and $|\beta_{k+1}| \leq M'_{k+1} \cdot |B|$. □

Without any further work, we obtain the fact that $|\beta_{\gamma,b}| \leq M'_{N_0}|B|$, hence

$$|\beta| \leq M'_{N_0} \cdot |\Gamma_0| \cdot |B|^2,$$

showing indeed that $|\beta| = \mathcal{O}(|B|^2)$. *A priori*, this gives us the bound

$$|S_\varphi(B)| = \mathcal{O}(|B|^3).$$

With another application of Lemma 2.14, we can get the desired result.

Let

$$M'' := 3N_0N_1M'_{N_0} \cdot |\Psi| \cdot |\Gamma_0|.$$

Lemma 4.7. $|S_\varphi(B)| \leq M''|B|^2$.

Proof. As we did before Lemma 4.5, for each $\psi \in \Psi$ and $\bar{c} \in \beta$, let $D_{\psi,\bar{c}} \subseteq \mathcal{U}_u$ with $|D_{\psi,\bar{c}}| \leq N_1$ be such that, $\psi(x_1;\bar{c})$ is T-equivalent to a boolean combination of $\delta_1(x_1;d)$ for $d \in D_{\psi,\bar{c}}$. Let $D := \bigcup\{D_{\psi,\bar{c}} : \psi \in \Psi_{2^{N_0}}, \bar{c} \in \beta\}$. Since $|\beta| \leq M'_{N_0} \cdot |\Gamma_0| \cdot |B|^2$, we obtain

$$|D| \leq N_1 \cdot |\Psi| \cdot M'_{N_0} \cdot |\Gamma_0| \cdot |B|^2.$$

Let $q_0, \ldots, q_{m-1} \in S$ enumerate S and, for each $i < m$, choose $p_i \in S_{\delta_1}(D)$ such that $p_i \vdash q_i$. First, we claim that, for all $i < m-1$,

$$|\mathcal{V}(q_{i+1}, B) \setminus \mathcal{V}(q_i, B)| \leq N_0 \cdot \operatorname{dist}(p_i, p_{i+1}).$$

Fix $i < m-1$ and let

$$A = \{\langle \gamma, b, \bar{c}\rangle : \gamma \in \Gamma_0, b \in B, \bar{c} \in \beta_{\gamma,b}\}, \text{ and}$$
$$A' = \{\langle \gamma, b, \bar{c}\rangle \in A : (\exists d \in D_{\psi,b^\frown\bar{c}})\,[\langle d, \delta_1\rangle \in \operatorname{diff}(p_i, p_{i+1})]\}$$

Clearly, $|A'| \leq \operatorname{dist}(p_i, p_{i+1})$.

For $\langle \gamma, b, \bar{c}\rangle \in A$, we define $\mathcal{V}^0_{\gamma,b,\bar{c}}$ and $\mathcal{V}^1_{\gamma,b,\bar{c}}$ as follows: Fix $a_1 \models q_i$, and $a'_1 \models q_{i+1}$. Fix $n, n' \leq N_0$, $d_0, \ldots, d_{n-1} \in \mathcal{U}_z$, and $d'_0, \ldots, d'_{n'-1} \in \mathcal{U}_z$ such that $\{\delta_0(x_0; d_j) : j < n\}$ is the set of components of $\gamma(x_0; a_1, b)$ and $\{\delta_0(x_0; d'_j) : j < n'\}$ is the set of components of $\gamma(x_0; a'_1, b)$. Let $\mathcal{V}^0_{\gamma,b,\bar{c}} = \{\nu_{a_1,d_j} : j < n\}$ and $\mathcal{V}^1_{\gamma,b,\bar{c}} = \{\nu_{a'_1,d'_j} : j < n'\}$.

For all $\langle \gamma, b, \bar{c}\rangle \in A \setminus A'$, we have $\mu_{q_i,\gamma,b,\bar{c}} = \mu_{q_{i+1},\gamma,b,\bar{c}}$. In other words, for all such $\langle \gamma, b, \bar{c}\rangle$, q_i and q_{i+1} decide the same generic Γ_0-types of $\gamma(x_0; x_1, b)$ over $\bar{c}$. Thus, $\mathcal{V}^1_{\gamma,b,\bar{c}} = \mathcal{V}^0_{\gamma,b,\bar{c}}$ and $|\mathcal{V}^1_{\gamma,b,\bar{c}} \setminus \mathcal{V}^0_{\gamma,b,\bar{c}}| = 0$. For all $\langle \gamma, b, \bar{c} \rangle \in A'$, clearly $|\mathcal{V}^1_{\gamma,b,\bar{c}} \setminus \mathcal{V}^0_{\gamma,b,\bar{c}}| \leq n' \leq N_0$. Moreover, by construction (and the proof of Lemma 4.4),

$$\mathcal{V}(q_{i+1}, B) \setminus \mathcal{V}(q_i, B) \subseteq \bigcup_{\langle \gamma, b, \bar{c} \rangle \in A} \left(\mathcal{V}^1_{\gamma,b,\bar{c}} \setminus \mathcal{V}^0_{\gamma,b,\bar{c}} \right).$$

Hence,

$$|\mathcal{V}(q_{i+1}, B) \setminus \mathcal{V}(q_i, B)| \leq N_0 \cdot \operatorname{dist}(p_i, p_{i+1}).$$

Since

$$S_{\Gamma_0}(B) \subseteq \bigcup_{i<m} \mathcal{V}(q_i, B),$$

Lemma 2.14 yields

$$|S_{\Gamma_0}(B)| \leq |\mathcal{V}(q_0, B)| + N_0(2 \cdot |D| + 2),$$

so

$$|S_{\Gamma_0}(B)| \leq N_0 \cdot |\Gamma_0| \cdot |B| + 1 + 2N_0 N_1 M'_{N_0} \cdot |\Psi| \cdot |\Gamma_0| \cdot |B|^2.$$

Since $|S_\varphi(B)| \leq |S_{\Gamma_0}(B)|$, we get

$$|S_\varphi(B)| \leq M''|B|^2.$$

□

As M'' did not depend on our choice of B, we get $|S_\varphi(B)| = \mathcal{O}(|B|^2)$. As $\varphi(x; y)$ was arbitrary with $|x| = 2$, this shows that $\pi_T(2) = 2$, which concludes the proof of Theorem 1.4.

4.4. **Conclusion.** Although it may be tempting to suppose that, using induction with the above proof, we should be able to get $\pi_T(n) = n$, this does not work with the current framework. It is vital that both x_0 and x_1 be singletons in the above argument. The reason that induction works in Section 3 is because φ-types correspond to generic types of balls over the same set in question. To describe a $\varphi(x; y)$-type over B, one needs only $|x|$ elements *from* B. However, in the general argument above, $\beta \subseteq B^{2^{N_0}+1}$. In order to bound the size of β (e.g., in Lemma 4.6), we need to know *a priori* that x_1 is a singleton. Still, it is the hope of this author that some modification of this proof will provide a positive answer to Open Question 1.3.

References

[1] H. Adler, *Theories controlled by formulas of Vapnik-Chervonenkis codimension 1*, Preprint (2008).

[2] A. Aschenbrenner, A. Dolich, D. Haskell, H. D. MachPherson, and S. Starchenko, *Vapnik-Chervonenkis density in some theories without the independence property, I*, Trans. Amer. Math. Soc. **368** (2016), 5889-5949.

[3] ———, *Vapnik-Chervonenkis density in some theories without the independence property, II*, Notre Dame J. Formal Logic **54** (2013), 311-363.

[4] S. Basu and D. Patel, *VC density of definable families over valued fields*, J. Eur. Math. Soc. to appear.

[5] S. Cotter and S. Starchenko, *Forking in VC-minimal theories*, J. Symbolic Logic **77** (2012), 1257-1271.

[6] J. Flenner and V. Guingona, *Canonical forests in directed families*, Proc. Amer. Math. Soc. **142** (2014), 1849-1860.

[7] ———, *Convexly orderable groups and valued fields*, J. Symbolic Logic **79** (2014), 154-170.

[8] V. Guingona and C. D. Hill, *On VC-Density Over Indiscernible Sequences*, Math. Logic Quart. **60** (2014), 59-65.

[9] V. Guingona and M. C. Laskowski, *On VC-Minimal Theories and Variants*, Arch. Math. Logic **52** (2013), no. 7, 743-758.

[10] H. R. Johnson and M. C. Laskowski, *Compression schemes, stable definable families, and o-minimal structures*, Discrete Comput. Geom. **43** (2010), no. 4, 914–926.

[11] I. Kaplan, A. Onshuus, and A. Usvyatsov, *Additivity of the dp-rank*, Trans. Amer. Math. Soc. **365** (2013), 5783-5804.

[12] M. C. Laskowski, *Vapnik-Chervonenkis classes of definable sets*, J. London Math. Soc. **45** (1992), no. 2, 377–384.

Towson University
URL: `https://tigerweb.towson.edu/vguingona/`
Email address: `vguingona@towson.edu`