Multivariate C^1-continuous splines on the Alfeld split of a simplex

Alexei Kolesnikov and Tatyana Sorokina

Abstract Using algebraic geometry methods and Bernstein-Bézier techniques we find the dimension of C^1-continuous splines on the Alfeld split of a simplex in $\mathbb{R}^n$ and describe a minimal determining set for this space.

1 Introduction

Let $\mathcal{P}^n$ denote the set of polynomials in n-variables over $\mathbb{R}$. In approximation theory, a spline is a piecewise-polynomial function defined on a polyhedral domain $\Omega \subset \mathbb{R}^n$ that belongs to a certain smoothness class. More precisely, for a fixed partition Δ of the domain Ω into a finite number of n-dimensional polyhedral subsets σ, a spline space is defined with respect to that partition:

$$S_d^r(\Delta) = \{s \in C^r(\Omega) : s|_\sigma \in \mathcal{P}_d^n \text{ for all } \sigma \in \Delta\}.$$

We use the following notation

$$S^r(\Delta) := \bigcup_{d \geq 0} S_d^r, \quad S(\Delta) := \bigcup_{r \geq 0} S^r(\Delta).$$

The Bernstein-Bézier techniques have become a standard tool used to analyze multivariate splines. We assume that the reader is familiar with the concepts of domain points, rings, disks, determining sets, and smoothness conditions, see [7], [3], [2].

To explain the approach of algebraic geometry, let us temporarily suspend the smoothness assumption. A piecewise polynomial function can be written as $s =$

Alexei Kolesnikov
Towson University, 7800 York Road, Towson, MD 21252 e-mail: `akolesnikov@towson.edu`

Tatyana Sorokina
Towson University, 7800 York Road, Towson, MD 21252 e-mail: `tsorokina@towson.edu`

$\sum_{\sigma\in\Delta} p_\sigma \cdot \chi_\sigma$, where χ_σ is the characteristic function of the set σ and $p_\sigma \in \mathcal{P}^n$. Thus, the set $S(\Delta)$ can be naturally identified with a subset of the following set:

$$\mathcal{R}_n(\Delta) := \Big\{ \{(\sigma, p_\sigma)\}_{\sigma\in\Delta} :\ p_\sigma \in \mathcal{P}^n \Big\}.$$

The set $\mathcal{R}_n(\Delta)$ has the natural structure of a module over the ring $\mathcal{P}^n$: the sum and scalar multiplication are defined as follows:

$$\{(\sigma, p_\sigma)\}_{\sigma\in\Delta} + \{(\sigma, q_\sigma)\}_{\sigma\in\Delta} := \{(\sigma, p_\sigma + q_\sigma)\}_{\sigma\in\Delta},$$
$$p \cdot \{(\sigma, p_\sigma)\}_{\sigma\in\Delta} := \{(\sigma, p \cdot p_\sigma)\}_{\sigma\in\Delta}.$$

Let $\mathcal{R}_n^r(\Delta)$ be the subset of $\mathcal{R}_n(\Delta)$ that corresponds to $S^r(\Delta)$. This subset is easily seen to be a submodule of $\mathcal{R}_n(\Delta)$. Let σ and σ' be adjacent regions, that is, the regions sharing an $(n-1)$-dimensional face or *facet* $\sigma \cap \sigma'$ located on the hyperplane with the equation $l_{\sigma\cap\sigma'} = 0$. Then the smoothness condition of order r across the facet is given by the smoothness equation

$$p_\sigma - p_{\sigma'} = l_{\sigma\cap\sigma'}^{r+1} \cdot q_{\sigma,\sigma'},$$

for some polynomial $q_{\sigma,\sigma'}$. The key idea behind our approach can be phrased as follows: if two different partitions of Ω give rise to the same set of equations, then the spaces of splines of degree $\leq d$ for the two partitions are isomorphic. For a more detailed treatment of spline modules we refer the reader to [8] and [5].

The paper is organized as follows. In Section 2 we introduce the Alfeld split A_n of a simplex T^n in $\mathbb{R}^n$, and the associated Alfeld pyramid $\hat{A}_n$. We prove that the space of splines $S_d^r(A_n)$ on the Alfeld split is isomorphic to the space of splines $S_d^r(\hat{A}_n)$ on the Alfeld pyramid. In Section 3 we construct a determining set for $S_d^1(A_n)$. In Section 4, we find the dimension of $S_d^1(\hat{A}_n)$ using induction on the spacial dimension n. Since $S_d^1(\hat{A}_n)$ and $S_d^1(A_n)$ are isomorphic they have the same dimension. We conclude the paper with several remarks in Section 5.

2 Splines on the Alfeld split and pyramid

Let $T^n := [v_1, \ldots, v_{n+1}]$ be a non-degenerate simplex in $\mathbb{R}^n$, and A_n be its Alfeld split around an interior point v_0 into $n+1$ subsimplices, see Fig. 1 and Fig. 2 for the two-dimensional case. We note that the two-dimensional Alfeld split coincides with the Clough–Tocher split, and few authors refer to the n-dimensional Alfeld split as the Clough–Tocher split as well. This is inaccurate since there exists an n-dimensional Clough–Tocher split, different from the Alfeld split, see [9] and references therein. Each subsimplex in the Alfeld split is a convex hull of a facet of T^n and v_0. We index the subsimplices of the split as follows. The simplex $\sigma_i := [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}]$ is the unique n-simplex opposite v_i. The common facet of the simplices σ_i and σ_j, $i < j$, will be denoted $\tau_{i,j}$. We may assume that v_0

is the origin, $v_1 = -\sum_{i=1}^n e_i$ and $v_{i+1} = e_i$ for $i = 1,\dots,n$, where e_i is the standard basis vector in $\mathbb{R}^n$. It is immediate to check that for $1 \le i \le n$ the facet $\tau_{1,i+1}$ lies on the hyperplane $x_i = 0$. For a pair (i, j), where $1 \le i \le n-1$ and $i+1 \le j \le n$, the facet $\tau_{i+1,j+1}$ lies on the hyperplane $x_i - x_j = 0$.

In this section we show that the space of splines of a given polynomial degree d and smoothness r on the Alfeld split is isomorphic to the space of splines over a different partition of T^n that we call the *Alfeld pyramid*. In the rest of the paper, we compute the dimension of the spline space on the Alfeld pyramid split using the Bernstein-Bézier methods and induction on the spacial dimension n. Given the Alfeld split A_n described above, the associated Alfeld pyramid $\widehat{A}_n$ is the partition of T^n into n simplices $\{\widehat{\sigma}_i\}_{i=2}^{n+1}$ and one non-convex polytope $\widehat{\sigma}_1$. For $i = 2,\dots,n+1$, the simplex $\widehat{\sigma}_i$ has the vertices

$$\{v_0, u_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_{n+1}\}, \text{ where } u_1 := -\frac{v_1}{n}, \tag{1}$$

and the polytope $\widehat{\sigma}_1$ is $T^n \setminus (\mathbb{R}^+)^n$. Fig. 3 shows the Alfeld pyramid split in the two-dimensional case. Denoting by $\widehat{\tau}_{i,j}$ the $(n-1)$-simplex which is a common facet of $\widehat{\sigma}_i$ and $\widehat{\sigma}_j$, we note that for each pair (i, j), the facets $\tau_{i,j}$ and $\widehat{\tau}_{i,j}$ lie on the same hyperplane. This is the key property connecting the Alfeld split A_n and the Alfeld pyramid $\widehat{A}_n$. We denote by P_n the collection of simplices $\widehat{\sigma}_i$, $i = 2,\dots,n+1$. This collection is a subset of the Alfeld pyramid, see Fig. 4 for the the two-dimensional case.

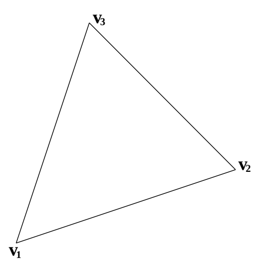

Fig. 1 Simplex T^2

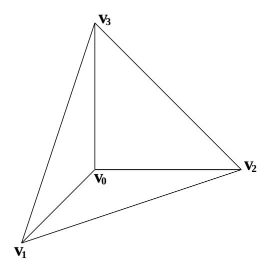

Fig. 2 Alfeld split A_2

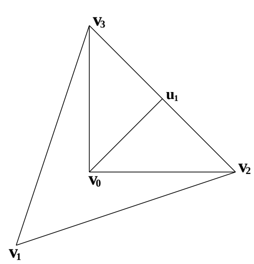

Fig. 3 Alfeld pyramid $\hat{A}_2$

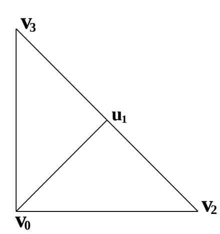

Fig. 4 Pyramid P_2

Theorem 1. *For all $n \geq 2$, and for all $d, r \geq 0$, the spline spaces $S_d^r(A_n)$ and $S_d^r(\widehat{A}_n)$ are isomorphic. In particular,*

$$\dim S_d^r(A_n) = \dim S_d^r(\hat{A}_n).$$

Proof. Following [4] or [8], we treat the spline module $\mathcal{R}^r(A_n)$ as the projection onto the first $n+1$ coordinates of the syzygy module of the system of column vectors in the following matrix:

$$\begin{bmatrix} \delta_{(1,2),1} & \dots & \delta_{(1,2),n+1} & x_1^{r+1} & 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \delta_{(1,n+1),1} & \dots & \delta_{(1,n+1),n+1} & 0 & \dots & x_n^{r+1} & 0 & \dots & 0 \\ \delta_{(2,3),1} & \dots & \delta_{(2,3),n+1} & 0 & \dots & 0 & (x_1-x_2)^{r+1} & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \ddots & \ddots & \vdots \\ \delta_{(n,n+1),1} & \dots & \delta_{(n,n+1),n+1} & 0 & \dots & 0 & \dots & 0 & (x_n-x_{n+1})^{r+1} \end{bmatrix},$$

where for $1 \leq i < j \leq n+1$ and $1 \leq k \leq n+1$,

$$\delta_{(i,j),k} = \begin{cases} 0, & \text{if } k \notin \{i,j\}, \\ (-1)^{k+j}, & \text{if } k = i, \\ (-1)^{i+k+1}, & \text{if } k = j. \end{cases}$$

It remains to note that the matrix associated with the Alfeld pyramid $\widehat{A}_n$ is exactly the same as the one above. Thus the modules $\mathcal{R}^r(A_n)$ and $\mathcal{R}^r(\widehat{A}_n)$ are equal. Since

$$S_d^r(A_n) \cong \{(p_1,\dots,p_{n+1}) \in \mathcal{R}^r(A_n) |\ deg(p_i) \leq d \ \ \forall i = 1,\dots,n+1\},$$

and

$$S_d^r(\widehat{A}_n) \cong \{(p_1,\dots,p_{n+1}) \in \mathcal{R}^r(\widehat{A}_n) |\ deg(p_i) \leq d \ \ \forall i = 1,\dots,n+1\},$$

the result follows. □

3 A determining set for $S_d^1(A_n)$

We recall that given any simplex T^n in $\mathbb{R}^n$, every polynomial p of degree $\leq d$ can be written uniquely in the form

$$p = \sum_{i_1+\dots+i_{n+1}=d} c_{i_1\dots i_{n+1}} B^d_{i_1\dots i_{n+1}}, \tag{2}$$

where $B^d_{i_1\dots i_{n+1}}$ are the Bernstein basis polynomials associated with T^n. As usual, we call the $c_{i_1\dots i_{n+1}}$ the B-coefficients of p, and define the associated domain point as

$$\xi^d_{i_1\dots i_{n+1}} := (i_1 v_1 + \cdots + i_{n+1} v_{n+1})/d, \quad i_1 + \cdots + i_{n+1} = d. \tag{3}$$

The point $\xi_{i_1\dots i_{n+1}}$ is at distance l from the face $[v_1,\dots,v_k]$ if $i_1+\cdots+i_k \ge d-l$. A ring $R_l(v_0)$ of raduis l around v_0 is the set of domain points at distance l from v_0. The disk $D_l(v_0)$ is the union of rings of radius $\le l$ around v_0. Distances, rings and disks associated with other faces of T^n are defined similarly. Given Δ, every spline $s \in S^0_d(\Delta)$ can be associated with the set of B-coefficients of its polynomial pieces, and with the set $\mathcal{D}_{d,\Delta}$ of the domain points corresponding to those coefficients.

We begin this section with two simple combinatorial facts. The proofs are based on the tools from the Bernstein-Bézier analysis in order to facilitate the transition to the domain point count in the subsequent theorems.

Lemma 1. *For positive integers n and m, let*

$$\mathcal{I}^n_m := \{(i_1,\dots,i_{n+1}) \in \mathbb{Z}^{n+1} \mid i_j \ge 0,\ \forall\ j \in \{1,\dots,n+1\},\ \sum_{j=1}^{n+1} i_j = m, \},$$
$$\mathcal{M}^n_m := \{(i_1,\dots,i_{n+1}) \in \mathcal{I}^n_m \mid \exists \text{ a unique } j \in \{1,\dots,n+1\} \text{ with } i_j = 0\}.$$

Then $|\mathcal{M}^n_m| = (n+1)\binom{m-1}{n-1}$.

Proof. Fix $j \in \{1,\dots,n+1\}$ and set $i_j = 0$. Then $|\mathcal{M}^n_m| = (n+1)|\mathcal{J}^n_m|$, where

$$\mathcal{J}^n_m := \{(i_1,\dots,i_n) \in \mathbb{Z}^n \mid \quad \forall\ j \in \{1,\dots,n\},\ i_j > 0, \quad i_1+\cdots+i_n = m\}.$$

In the Bernstein-Bézier analysis, $|\mathcal{J}^n_m|$ is the number of the domain points of a polynomial of degree $\le m$ in $(n-1)$ variables that are strictly interior to the $(n-1)$-simplex. This number is $\binom{m-1}{n-1}$. Thus, $|\mathcal{M}^n_m| = (n+1)\binom{m-1}{n-1}$. □

Lemma 2. *Let $\mathcal{I}^n_m$ be as in Lemma 1. Suppose*

$$\mathcal{N}^n_m := \{(i_1,\dots,i_{n+1}) \in \mathcal{I}^n_m \mid \quad \exists \ j \in \{1,\dots,n+1\} \quad \text{such that} \quad i_j = 0\}.$$

Then $|\mathcal{N}^n_m| = \binom{m+n}{n} - \binom{m-1}{n}$.

Proof. In the Bernstein-Bézier analysis this is the number of the domain points of a polynomial of degree $\le m$ in n variables that are on the boundary of the n-simplex. The easiest way to compute it is to subtract the number of the domain points that are strictly interior to the n-simplex from the total number of the domain points of a polynomial of degree $\le m$ in n variables in the n-simplex. □

Consider a spline $s \in S^1_d(A_n)$. The set of all B-coefficients associated with this spline is the union of $(n+1)$ sets of B-coefficients associated with the polynomials $s|_{\sigma_i}$, $i = 1,\dots,n+1$, on each subsimplex. Accordingly, the set of all the domain points associated with s is the union of the domain points for each of the $s|_{\sigma_i}$ in σ_i. One of the key ideas in the argument below is to organize the domain points as

$$\mathcal{D} := \bigcup_{m=0}^{d} R_m(v_0) = \bigcup_{m=0}^{d} \xi^m_I, \quad I \in \mathcal{N}^n_m,$$

where each ξ_I^m is as in (3), $R_m(v_0)$ is the ring of radius m around v_0, and $\mathcal{N}_m^n$ is as in Lemma 2. Indeed, for any $0 \le m \le d$, the ring $R_m(v_0)$ of radius m around v_0 is exactly the set of domain points on the boundary of an n-simplex $T_m^n := [mv_1/d, \ldots, mv_{n+1}/d]$. Note that these domain points are the boundary domain points associated with a single polynomial of degree m defined on the simplex T_m^n. This notation establishes a one-to-one correspondence between each domain point and a pair (m, I), where $m \in \{0, \ldots, d\}$ and $I \in \mathcal{N}_m^n$.

We need two basic facts from the Bernstein-Bézier analysis.

Lemma 3. *Let $s \in S_d^1(A_n)$, and let $\mathcal{I}_m^n$ be as in Lemma 1. Suppose*

$$\mathcal{F}_m^n := \{(i_1, \ldots, i_{n+1}) \in \mathcal{I}_m^n | \quad \exists\, j, k \in \{1, \ldots, n+1\} \\ \textit{such that} \;\; j \neq k \;\; \textit{and} \;\; i_j = i_k = 0\}.$$

Then for each $0 \le m < d$, the coefficient $c_I^m \in R_m(v_0)$, where $I \in \mathcal{F}_m^n$, can be determined as a linear combination $\mathcal{L}$ of the following $n+1$ coefficients located on $R_{m+1}(v_0)$

$$c^m_{i_1,\ldots,i_{n+1}} = \mathcal{L}(c^{m+1}_{i_1+1,\ldots,i_{n+1}}, c^{m+1}_{i_1,i_2+1\ldots,i_{n+1}}, \ldots, c^{m+1}_{i_1,\ldots,i_{n+1}+1}). \tag{4}$$

Proof. This is a rewrite of the usual smoothness conditions across interior faces of A_n. Indeed, without loss of generality assume $i_1 = i_2 = \cdots = i_k = 0$, $k \ge 2$. Then ξ_I^m lies on the interior face $F_k := [v_0, v_{k+1}, \ldots, v_{n+1}]$ shared by k subsimplices in A_n of the form

$$\sigma_j = [v_0, v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k, v_{k+1}, \ldots, v_{n+1}], \quad j = 1, \ldots, k.$$

Then, we apply C^1 smoothness conditions across F_k to any two such subsimplices. Each smoothness functional combines c_I^m, and the $n+1$ coefficients associated with the domain points on T_{m+1}^n that are located at distance one from ξ_I^m, in a linear equation and yields (4). □

The proof of the next result can be found in Theorem 6.3 of [10].

Lemma 4. *Let $s \in S_d^1(A_n)$. Then $s \in C^n(v_0)$.*

Combining Lemma 4 and Theorem 1 we obtain the following

Lemma 5. *Let $s \in S_d^1(\hat{A}_n)$. Then $s \in C^n(v_0)$.*

We note that in both lemmas above, $C^n(v_0)$ is understood in the sense of equality of all partial derivatives of order up to n at v_0. We are now ready to construct a determining set for $S_d^1(A_n)$ as in Definition 5.12 of [7], that is a subset $\mathcal{G}$ of $\mathcal{D}_{d,\Delta}$ such that if $s \in S_d^1(A_n)$ and the B-coefficients corresponding to all domain points in $\mathcal{G}$ vanish, then s vanishes as well. Note that at this point we do not prove that this determining set is minimal.

Theorem 2. *Let $s \in S_d^1(A_n)$. Then the following is a determining set.*

$$DS_d^1(A_n) := \{\xi_I^d, \;\; I \in \mathcal{N}_d^n\} \cup \{\xi_I^m, \;\; j \in \{n, \ldots, d-1\}, \;\; I \in \mathcal{M}_m^n\},$$

and $|DS^1_d(A_n)| = \binom{d+n}{n} + n\binom{d-1}{n}$. *This set consists of all domain points on the exterior of* T^n*, and the domain points strictly interior to each boundary facet of every simplex* T^n_m *for* $m = d-1,\ldots,n$.

Proof. It suffices to show that if all coefficients of s corresponding to $DS^1_d(A_n)$ are set to zero, then $s \equiv 0$. We start with setting to zero all the coefficients associated with $\{\xi^d_I,\ I \in \mathcal{N}^n_d\}$, or, equivalently, on $R_d(v_0)$. From Lemma 2 we obtain

$$|\{\xi^d_I,\ I \in \mathcal{N}^n_d\}| = \binom{d+n}{n} - \binom{d-1}{n}.$$

Next we move to $R_{d-1}(v_0)$. All indices I on this ring have two properties: the number of elements in I is $d-1$, and at least one entry in I is zero. According to Lemma 3, each coefficient corresponding to I with two or more zero entries vanishes. Thus, we only need to set to zero the coefficients corresponding to I with precisely one zero entry. We repeat this process for each m between $d-1$ and n. That is, once $R_{m+1}(v_0)$ is populated with zeros, we move to $R_m(v_0)$, where all indices I have two properties: the number of elements in I is m, and at least one entry in I is zero. According to Lemma 3, each coefficient corresponding to I with two or more zero entries vanishes. Thus, we only need to set to zero the coefficients corresponding to I with precisely one zero entry. From Lemma 1 we obtain

$$|\{\xi^m_I,\ I \in \mathcal{M}^n_m\}| = (n+1)\binom{m-1}{n-1},\ \ m \in \{d-1, d-2\ldots,n\}.$$

When we populate $R_n(v_0)$ with zeros, we note that by Lemma 4, the disk $D_n(v_0)$ can be considered as a single simplex since the coefficients of s corresponding to this disk form a polynomial of degree n. We note that from Lemma 2, the total number of domain points on $R_n(v_0)$ is $\binom{2n}{n}$ which is precisely the dimension of polynomials degree $\le n$ in n variables. Moreover, this polynomial of degree n vanishes on all $n+1$ faces of the simplex T^n_n, and thus it is a zero polynomial. Therefore, all coefficients of s in $D_n(v_0)$ vanish. We now do the final count

$$\begin{aligned}|DS^1_d(A_n)| &= \binom{d+n}{n} - \binom{d-1}{n} + (n+1)\sum_{m=n}^{d-1}\binom{m-1}{n-1}\\ &= \binom{d+n}{n} - \binom{d-1}{n} + (n+1)\binom{d-1}{n} = \binom{d+n}{n} + n\binom{d-1}{n}.\end{aligned}$$

The proof is complete. We note that if the dimension of $S^1_d(A_n)$ is known to be $|DS^1_d(A_n)|$ then $DS^1_d(A_n)$ would be a minimal determining set. $\square$

4 The main result

In this section we compute the dimension of C^1-continuous splines defined over the Alfeld pyramid $\hat{A}_n$ in $\mathbb{R}^n$. By Theorem 1 this is equal to the dimension of C^1-continuous splines over the Alfeld split A_n of a single simplex in $\mathbb{R}^n$. We note that from Theorem 9.3 in [7] it follows that $\dim S^1_d(A_2) = \binom{d+2}{2} + 2\binom{d-1}{2}$. In Remark 6, we illustrate the idea of our proof for $n = 2$ since this is the only case with clear visual illustration.

Theorem 3. *For all integers $d \geq 0$ and $n \geq 1$,*

$$\dim S^1_d(A_n) = \binom{d+n}{n} + n\binom{d-1}{n}.$$

Proof. We use induction on n. Since A_1 is the split of a line segment into two subsegments, it is immediate that $\dim S^1_d(A_1) = 2d$.

For $n \geq 2$, in view of Theorem 1 we can consider $S^1_d(\hat{A}_n)$ instead of $S^1_d(A_n)$. The dimension of $S^1_d(\hat{A}_n)$ is equal to the dimension of polynomials of degree $\leq d$ in n variables $\binom{d+n}{n}$ plus the dimension of

$$S_0 := \{s \in S^1_d(\hat{A}_n) |\ \ s \equiv 0 \text{ everywhere outside of the pyramid } P_n\}.$$

We treat S_0 as a subspace of $S^1_d(P_n)$. The plan is to use the induction hypothesis to compute the dimension of $S^1_d(P_n)$ and then subtract the number of domain points accosiated with vanishing B-coefficients due to the condition $s \equiv 0$ outside of P_n. We recall that the pyramid P_n is the split of the simplex $[v_0, v_2, \ldots, v_{n+1}]$ into n subsimplices with the split point $u_1 := -v_1/n$, as in (2). The domain points inside P_n are located on the union of rings $R_i(v_0)$, $i = 0, \ldots, d$. These rings lie on parallel $(n-1)$-simplices $T^{n-1}_i := [iv_2/d, \ldots, iv_{n+1}/d]$. Each simplex T^{n-1}_i is partitioned as $(n-1)$-dimensional Alfeld split A^i_{n-1} by the point iu_1/d. Therefore, the domain points in the pyramid P_n can be considered as the domain points for the Alfeld splits A^i_{n-1} of T^{n-1}_i. Moreover, since all T^{n-1}_i are parallel in $\mathbb{R}^n$, all C^1 smoothness conditions across interior faces of P_n are those for for the $(n-1)$-dimensional Alfeld split. Thus, using the induction hypothesis on A^i_{n-1} , we obtain

$$\dim S^1_d(P_n) = \sum_{i=0}^{d} \dim S^1_i(A^i_{n-1}) = \sum_{i=0}^{d} \left[\binom{i+n-1}{n-1} + (n-1)\binom{i-1}{n-1} \right].$$

Moreover, the induction hypothesis along with Theorem 2 provides *minimal* determining sets $DS^1_i(A^i_{n-1})$. In order to find the dimension of S_0 we need to know the number N_i of points in $DS^1_i(A^i_{n-1})$ that have associated vanishing B-coefficients after joining the zero function outside of P_n. Then

$$\dim S_0 = \sum_{i=0}^{d} \dim\,(S^1_i(A^i_{n-1}) - N_i). \tag{5}$$

We now compute N_i. Due to the supersmoothness result of Lemma 5, any $s \in S_0$ is $C^n(v_0)$. Thus

$$N_i = \binom{i+n-1}{n-1} = \dim S_i^1(A_{n-1}^i) \quad \text{for} \quad 0 \le i \le n.$$

For each $i > n$, the C^1 smoothness conditions across the boundary of P_n affect the coefficients associated with the domain points located on the boundary and one layer inside of P_n. More precisely, they are located in the rings $R_i(iu_1/d)$ and $R_{i-1}(iu_1/d)$. They form a subset of $DS_i^1(A_{n-1}^i)$. Lemma 1, Lemma 2 and Theorem 2 provide the complete description and the number of such domain points:

$$DS_i^1(A_{n-1}^i) \cap R_i(iu_1/d) = \{\xi_I^i,\ I \in \mathcal{N}_i^{n-1}\},$$
$$DS_i^1(A_{n-1}^i) \cap R_{i-1}(iu_1/d) = \{\xi_I^{i-1},\ I \in \mathcal{M}_{i-1}^{n-1}\},$$

and

$$N_i = \binom{i+n-1}{n-1} - \binom{i-1}{n-1} + n\binom{i-2}{n-2}. \tag{6}$$

Substituting (6) into (5) we obtain

$$\dim S_0 = n \sum_{i=n+1}^{d} \left[\binom{i-1}{n-1} - \binom{i-2}{n-2}\right] = n \sum_{i=n+1}^{d} \binom{i-2}{n-1}$$
$$= n \sum_{i=0}^{d-n-1} \binom{i+n-1}{i} = n\binom{d-1}{n}.$$

Finally,

$$\dim S_d^1(A_n) = \dim S_d^1(\hat{A}_n) = \binom{d+n}{n} + \dim S_0 = \binom{d+n}{n} + n\binom{d-1}{n}.$$

The proof is now complete. □

5 Remarks

Remark 1. Theorem 2 combined with Theorem 3 provides a minimal determining set. This set can be used directly to construct C^1-continuous macro-elements based on the Alfeld split of a simplex. However, the polynomial degree d of such macro-elements is at least $2^{n-1}+1$. Thus for $n \ge 3$, without additional supersmoothness conditions, C^1-continuous macro-elements on the Alfeld split of a simplex have excessive number of free parameters and are hard to implement. For the case $n = 3$, additional smoothness is introduced in [1].

Remark 2. The work on finding dimensions of spline spaces $S_d^r(A_n)$ for higher values of r is in progress. The main difficulty is that the analog of Lemma 4 for $r > 1$ is not known. The lower bound for the supersmoothness at v_0 can be found in [10], but this bound is not exact. The supersmoothness at the split point is one of the main ingredients of the current proof of Theorem 3.

Remark 3. The conjecture on the dimension of $S_d^r(A_n)$ for all values of n, r, and d can be found in [6]. Our result in Theorem 3 proves this conjecture for $r = 1$, for all values of n and d.

Remark 4. There has been a considerable amount of work done with bivariate and trivariate macro-elements based on the Alfeld splits of a triangle and a tetrahedron, respectively. Such macro-element spaces can have dimension different from our result due to additional conditions imposed on them, see [1], [7] and references therein.

Remark 5. The minimal determining sets of Theorem 2 for $n = 2$ and $n = 3$ can be checked directly using P. Alfeld's software available on `http://www.math.utah.edu/~pa`. The software computes dimension of spline spaces for fixed vales of d as well.

Remark 6. In this remark we illustrate the idea of the proof of Theorem 3 for $n = 2$. We consider $S_d^1(\hat{A}_2)$ instead of $S_d^1(A_2)$, and refer to Fig. 2 and Fig. 3 to observe that

$$\dim S_d^1(\hat{A}_2) = \binom{d+2}{2} + \dim S_0, \quad \text{where}$$

$$S_0 := \{s \in S_d^1(\hat{A}_2) |\ s \equiv 0 \text{ everywhere outside of } [v_0, v_2, v_3]\}.$$

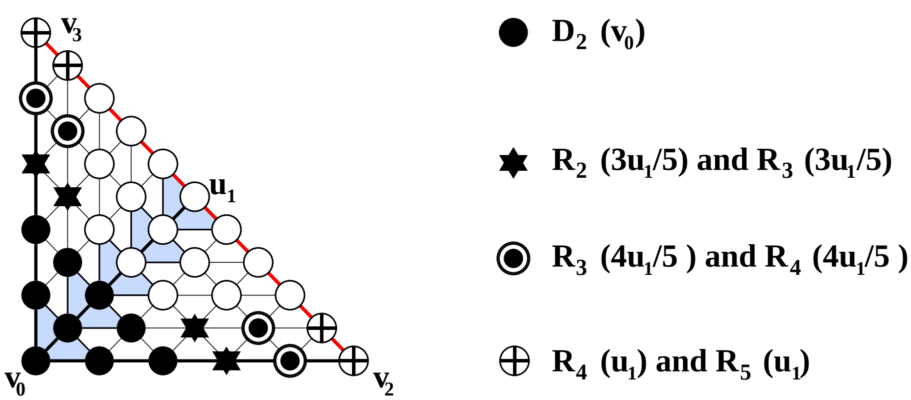

Fig. 5 Domain points in P_2 for $S_5^1(\hat{A}_2)$

The split of the triangle $[v_0, v_2, v_3]$ into two subtriangles, $[v_0, v_2, u_1]$ and $[v_0, v_3, u_1]$, forms the pyramid P_2. The domain points inside P_2 are located on the parallel line segments $T_i^1 := [iv_2/d, iv_3/d]$ partitioned into two subsegments $[iv_2/d, iu_1/d]$ and $[iv_3/d, iu_1/d]$ forming the Alfeld splits A_1^i. In Fig. 5, there are five segments T_i^1 split in half. Since all T_i^1 are parallel in $\mathbb{R}^2$, all C^1 smoothness conditions across $[v_0, u_1]$ are those for A_1^i. Each minimal determining set $DS_i^1(A_1^i)$, $i = 1, \ldots, d$, is formed by all domain points on $[iv_2/d, iv_3/d]$ except iu_1/d. The minimal determining set for $DS_i^1(A_1^0)$ is just v_0. In order to find the dimension of S_0 we need to know N_i the number of points in $DS_i^1(A_1^i)$ that have associated vanishing coefficients after joining the zero function outside of $[v_0, v_2, v_3]$. Then

$$\dim S_0 = \sum_{i=0}^{d} \dim \left(S_i^1(A_1^i) - N_i\right).$$

Due to supersmoothness two at v_0, the B-coefficients associated with the domain points in $D_2(v_0)$ marked as black dots in Fig. 5 vanish. Thus $N_0 = 1$, $N_1 = 2$ and $N_2 = 4$. For each $i > 2$, the C^1 smoothness conditions across $[v_0, v_2]$ and $[v_0, v_3]$ affect the coefficients associated with the domain points on $[v_0, v_2] \cup [v_0, v_3]$ and one layer inside of P_2. For example, in Fig. 5, the coefficients associated with the stars, the dots in circles, and the crosses all vanish due to C^1 smoothness conditions across $[v_0, v_2]$ and $[v_0, v_3]$. Therefore, $N_i = 4$ for $i > 2$, and

$$\dim S_d^1(\hat{A}_2) = \binom{d+2}{2} + \sum_{i=3}^{d} (2i - 4) = \binom{d+2}{2} + 2\binom{d-1}{2}.$$

Acknowledgements The first author was partially supported by the NSF grant DMS-0901315.

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