On the Existence of Unions of Timed Scenarios

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Abstract. In earlier work it was shown that the union of two timed scenarios is not necessarily a timed scenario. We report on a comprehensive study of the union operation for scenarios. We identify a sufficient condition for the existence of the union of two scenarios, and prove that the condition is also necessary. The condition provides a syntactic criterion for the existence of the union of two scenarios.

1 Introduction

Using scenarios for specifying complex systems (including real time systems and distributed systems [1, 2]), and synthesizing formal models of systems from scenarios have been active areas of research for several decades [3–10].

In our earlier work [11] we developed a formal, yet simple notation for timed scenarios. Intuitively, a scenario is a sequence of events along with a set of constraints between the times of these events, which can be used to specify the partial behaviours of a system or a component of a system (see Sec. 2 for more details). We defined the semantics of a timed scenario as the set of all behaviours that are "allowed" by the scenario. All such behaviours must satisfy the same set of constraints. We developed the notion of "stable distance table", as a *canonical* representation of the set of constraints of a scenario: it captures the tightest set of constraints of the scenario. We used stable distance tables as the foundation for various algorithms for determining the consistency and equivalency of scenarios, as well as for optimizing scenarios [11, 12]. More recently, we used them for developing the notions of intersection, union and subsumption for scenarios [13]. In particular, we studied the conditions under which the union of the behaviours allowed by two scenarios can be represented by a single scenario, and presented a sufficient condition for the existence of the union [13]. The problem of whether the condition is also necessary was left open [13].

The operations on scenarios are directly relevant to the problem of synthesizing timed automata from a set of scenarios [14]. These operations are, in general, also relevant to model-checking timed automata [15].

Model-checking [16] is a verification technique based on exhaustive exploration of the reachable state space of a system: given a model of a system and some property p, the goal is to determine whether there exists a state reachable from the initial state in which p holds. A major obstacle to using model-checking in practice is the state explosion problem. For real-time systems modeled as timed automata the state space is infinite due to clock variables, therefore a complete search of the state space is impossible. Various abstraction techniques have been developed to overcome this problem [17]. One such abstraction is using clock zones, as symbolic representations of the states of timed automata. A clock zone is a set of constraints, each of which puts a bound on the difference between the values of two clocks. Each location of a timed automaton is associated with a zone, and the reachability analysis involves manipulating clock zones along various paths of the automaton. A location, in general, can be associated with more than one zone, for example when it can be reached via different paths. In that case, the union of all these zones must be taken, in order to get the zone corresponding to the location. However, the union of zones, in general, cannot be represented by a zone, and therefore, it is often approximated [18, 17]. One such approximation is obtained by taking the convex hull of the zones [17].

Various data structures for representing zones, and a variety of reachability algorithms utilizing these data structures have been introduced [19, 17, 20]. Examples include using difference bound matrices (DBMs) [21], binary decision diagrams (BDDs) [18], clock difference diagrams (CDDs) [19] and Numerical decision digrams (NDDs) [22, 20].

As part of an earlier work [12, 23] we showed how—in the case of a timed automaton that corresponds to a single timed scenario—our distance tables can be used to represent time zones similarly to DBMs. So the next natural step for us is to investigate the necessary conditions under which it is possible to form the union of two timed scenarios represented by two distance tables. More specifically, we make the following contributions:

- We introduce the notion of *z_pair*s between two different constraints α and β of two scenarios ξ and η, respectively. Intuitively, α and β form a *z_pair*, if there exist behaviours whose times do not satisfy constraints α and β, but satify the constraints of the "quasi-union" of ξ and η (see Sec. 2.1 for details). That is, they do not belong to the semantics of either ξ or η, but belong to the semantics of the "quasi-union" of ξ and η. We call such behaviours "zigzagging behaviours". We show that if there is no *z_pair* between scenarios ξ and η, then $\xi \cup \eta$ exists. This will be another sufficient condition for the existence of the union of two scenarios, but stronger than our previous condition [13].
- We prove that the existence of *z_pair*s is also a necessary condition for the existence of zigzagging behaviours, and hence the non-existence of the union of ξ and η. This will address the problem that was previously left open [13].

2 Timed scenarios

This subsection briefly recounts our earlier work [11–13].

Let Σ be a finite set of symbols called *events*. A *behaviour*[1] over Σ is a sequence $(e_0, t_0)(e_1, t_1)(e_2, t_2)\dots$, such that $e_i \in \Sigma$, $t_i \in \mathbb{R}^{\geq 0}$ and $t_{i-1} \leq t_i$ for $i \in \{1, 2 \dots\}$. For a finite behaviour $\mathcal{B} = (e_0, t_0)(e_1, t_1)\dots(e_{n-1}, t_{n-1})$ of length

[1] The notion of "behaviour" is equivalent to that of Alur's "timed word" [15]. We found the term "behaviour" more suitable and intuitive in the context of timed scenarios.

n, and for any $0 \leq i < j < n$, the *distance*, in time units, of event j from event i in $\mathcal{B}$ is denoted by $t^{\mathcal{B}}_{ij}$. That is, $t^{\mathcal{B}}_{ij} = t_j - t_i$.

A *timed scenario* (*scenario* for short) of length $n \in \mathbb{N}$ over Σ is a pair $(\mathcal{E}, \mathcal{C})$, where $\mathcal{E} = e_o e_1 ... e_{n-1}$ is a sequence of events, and $\mathcal{C} \subset \Phi(n)$ is a finite set of constraints. Each constraint in $\Phi(n)$ is of the form $b \sim a$, where b is the symbol $\tau_{i,j}$ (for some integers $0 \leq i < j < n$), $\sim \in \{\leq, \geq\}$ and a is a constant in the set of rational numbers, $\mathbb{Q}$. The interpretation is that $\tau_{i,j}$ is the time distance between the i-th and the j-th events in the behaviours described by a scenario. The constraints $\tau_{i,j} \geq 0$ and $\tau_{i,j} \leq \infty$ are called *default constraints*.

A behaviour $\mathcal{B} = (e_0, t_0)(e_1, t_1) \dots (e_{n-1}, t_{n-1})$ over Σ is *allowed* by scenario $\xi = (\mathcal{E}, \mathcal{C})$ iff $\mathcal{E} = e_0 \dots e_{n-1}$ and every $\tau_{i,j} \sim a$ in $\mathcal{C}$ evaluates to true after $\tau_{i,j}$ is replaced by $t^{\mathcal{B}}_{ij}$. The *semantics* of scenario ξ, denoted by $[\![\xi]\!]$, is the set of behaviours that are allowed by ξ. A scenario ξ is *consistent* iff $[\![\xi]\!] \neq \emptyset$. It is *inconsistent* iff $[\![\xi]\!] = \emptyset$.

Fig. 1 shows the "external representation" of scenario $\xi = (abc, \{\tau_{0,1} \geq 3, \tau_{0,2} \leq 7\})$, $[\![\xi]\!] = \{(a, t_0)(b, t_1)(c, t_2) \mid t_0 \leq t_1 \leq t_2 \wedge t_1 - t_0 \geq 3 \wedge t_2 - t_0 \leq 7\}$.

For a consistent scenario ξ of length n, and for $0 \leq i < j < n$, $m^{\xi}_{ij} = min\{t^{\mathcal{B}}_{ij} \mid \mathcal{B} \in [\![\xi]\!]\}$ and $M^{\xi}_{ij} = max\{t^{\mathcal{B}}_{ij} \mid \mathcal{B} \in [\![\xi]\!]\}$[2]. For any behaviour $\mathcal{B}$ in $[\![\xi]\!]$, $0 \leq m^{\xi}_{ij} \leq t^{\mathcal{B}}_{ij} \leq M^{\xi}_{ij} \leq \infty$. Moreover, the following inequations hold [11]:

$$m^{\xi}_{ij} + m^{\xi}_{jk} \leq m^{\xi}_{ik} \leq \left\{ \begin{matrix} m^{\xi}_{ij} + M^{\xi}_{jk} \\ M^{\xi}_{ij} + m^{\xi}_{jk} \end{matrix} \right\} \leq M^{\xi}_{ik} \leq M^{\xi}_{ij} + M^{\xi}_{jk} \qquad (1)$$

Let $\xi = (\mathcal{E}, \mathcal{C})$ be a scenario of length n, such that, for any $0 \leq i < j < n$, $\mathcal{C}$ contains at most one constraint of the form $\tau_{i,j} \geq c$ and at most one of the form $\tau_{i,j} \leq c$. A *distance table*[3] for ξ is a representation of $\mathcal{C}$ in the form of a triangular matrix $\mathcal{D}^{\xi}$. For $0 \leq i < j < n$, $\mathcal{D}^{\xi}[i, j] = (l^{\xi}_{ij}, h^{\xi}_{ij})$, where l^{ξ}_{ij} and h^{ξ}_{ij} are rational numbers. If $\tau_{i,j} \geq c \in \mathcal{C}$ then $l^{\xi}_{ij} = c$, otherwise $l^{\xi}_{ij} = 0$; if $\tau_{i,j} \leq c \in \mathcal{C}$ then $h^{\xi}_{ij} = c$, otherwise $h^{\xi}_{ij} = \infty$. A distance table for ξ of Fig. 1 is shown in the middle of the figure.

We will write just l_{ij}, h_{ij}, m_{ij} and M_{ij} when ξ is understood.

A distance table for a scenario of size n is *valid* iff $l_{ij} \leq h_{ij}$, for all $0 \leq i < j < n$. A table that is not valid is *invalid*. If $\mathcal{D}^{\xi}$ is invalid, then ξ is inconsistent.

A valid distance table for a scenario of size n is *stable* iff, for all $0 \leq i < j < k < n$, the inequations in (1) hold when m_{ij}, m_{jk}, m_{ik} are replaced by l_{ij}, l_{jk}, l_{ik} and M_{ij}, M_{jk}, M_{ik} are replaced by h_{ij}, h_{jk}, h_{ik}. If $\mathcal{D}^{\xi}$ is stable then ξ is consistent. The table of a consistent scenario ξ can be stabilized by applying the following six rules:

$$l_{ij} + l_{jk} > l_{ik} \longrightarrow l_{ik} := l_{ij} + l_{jk} \qquad l_{ik} > l_{ij} + h_{jk} \longrightarrow l_{ij} := l_{ik} - h_{jk}$$
$$l_{ik} > h_{ij} + l_{jk} \longrightarrow l_{jk} := l_{ik} - h_{ij} \qquad l_{ij} + h_{jk} > h_{ik} \longrightarrow h_{jk} := h_{ik} - l_{ij}$$
$$h_{ij} + l_{jk} > h_{ik} \longrightarrow h_{ij} := h_{ik} - l_{jk} \qquad h_{ik} > h_{ij} + h_{jk} \longrightarrow h_{ik} := h_{ij} + h_{jk}$$

[2] The absence of an upper bound for some i and j will be denoted by $M^{\xi}_{ij} = \infty$.

[3] A detailed comparison with Dill's Difference Bounds Matrices (DBMs) [21] can be found in our earlier work [12].

At least one rule is applicable if and only if some inequation in (1) does not hold. The purpose of each rule is to tighten a constraint (i.e., increase a lower bound or decrease an upper bound) just enough to establish a particular inequation. If none of the rules is applicable, and the table is valid, then it is stable.

If ξ is a scenario of length n and $\mathcal{D}_s^\xi$ is its stable table, then for every $0 \le i < j < n$, $l_{ij}^\xi = m_{ij}^\xi$ and $h_{ij}^\xi = M_{ij}^\xi$, that is, $\mathcal{D}_s^\xi[i,j] = (m_{ij}^\xi, M_{ij}^\xi)$. Intuitively, $\mathcal{D}_s^\xi[i,j]$ corresponds to a pair of constraints: the time distance between events i and j in every behaviour in $[\![\xi]\!]$ must be at least m_{ij}^ξ and at most M_{ij}^ξ. We use $\mathcal{C}(\mathcal{D}_s^\xi)$ to denote the set of constraints represented by $\mathcal{D}_s^\xi$. All the constraints in $\mathcal{C}(\mathcal{D}_s^\xi)$ are as *tight* as possible. Moreover, $\mathcal{C}(\mathcal{D}_s^\xi)$ includes all the constraints that are implied [12] by the initial set of constraints.

A scenario ξ can be transformed to an equivalent scenario η by *optimizing* its set of constraints [23], so that η has a minimal set of constraints[4]. We call η the *optimized* form of ξ. For an optimized scenario $\xi = (\mathcal{E}, \mathcal{C})$ the members of $\mathcal{C}$ will be referred to as *explicit constraints*. We know that $\mathcal{D}_s^\xi$ also includes *implicit* constraints: default constraints and constraints that are implied by $\mathcal{C}$. The table on the right of Fig. 1 is the stable distance table of ξ in that figure. $\mathcal{D}_s^\xi$ represents the set of constraints $\{\tau_{0,1} \ge 3, \tau_{0,1} \le 7, \tau_{0,2} \ge 3, \tau_{0,2} \le 7, \tau_{1,2} \ge 0, \tau_{1,2} \le 4\}$. Observe that, for example, $\mathcal{D}_s^\xi[1,2] = (0,4)$: $m_{12}^\xi = 0$, which corresponds to a default constraint, and $M_{12}^\xi = 4$, which corresponds to an implied constraint. The set of explicit constraints of ξ, which is already in optimized form, is $\{\tau_{0,1} \ge 3, \tau_{0,2} \le 7\}$, while $\{\tau_{0,1} \le 7, \tau_{0,2} \ge 3, \tau_{1,2} \le 4\}$ is the set of implicit constraints.

The *interval* I_{ij}^ξ is $\{a \in \mathbb{Q} \mid m_{ij}^\xi \le a \le M_{ij}^\xi\}$ where $\mathcal{D}_s^\xi[i,j] = (m_{ij}^\xi, M_{ij}^\xi)$. Intuitively, I_{ij}^ξ specifies the set of all the possible values of t_{ij} that can appear in the behaviours allowed by ξ.

If $i = j$, then by definition, $t_{ii} = 0$, so $m_{ii} = 0$ and $M_{ii} = 0$.

2.1 Combination of two timed scenarios

If ξ and η are two consistent scenarios of length n with the same sequences of events, $\mathcal{E}$, such that $\forall_{0 \le i < j < n}\ I_{ij}^\xi \cap I_{ij}^\eta \ne \emptyset$, then the *combination* (or the "quasi-union") of ξ and η, denoted by $\xi \Cup \eta$, is *defined*. In that case, $\xi \Cup \eta$ is a scenario whose sequence of events is $\mathcal{E}$ and whose constraints are given by $\mathcal{D}^{\xi \Cup \eta}$, where $\mathcal{D}^{\xi \Cup \eta}[i,j] = (\min(m_{ij}^\xi, m_{ij}^\eta), \max(M_{ij}^\xi, M_{ij}^\eta))$.

If $\xi \Cup \eta$ is defined, then $[\![\xi]\!] \cup [\![\eta]\!] \subseteq [\![\xi \Cup \eta]\!]$. But, in general, $[\![\xi \Cup \eta]\!] \not\subseteq [\![\xi]\!] \cup [\![\eta]\!]$. This is because table $\mathcal{D}_s^{\xi \Cup \eta}$ allows all the behaviours in $[\![\xi]\!] \cup [\![\eta]\!]$, but there is a possibility that it may also allow some extra behaviours, namely those that satisfy all the constraints of the combination, but do not satisfy some of the constraints in ξ and some of the constraints in η. That is, $[\![\xi \Cup \eta]\!] = [\![\xi]\!] \cup [\![\eta]\!] \cup \mathcal{Z}(\xi,\eta)$, where $[\![\xi]\!] \cap \mathcal{Z}(\xi,\eta) = \emptyset$ and $[\![\eta]\!] \cap \mathcal{Z}(\xi,\eta) = \emptyset$. We call members of $\mathcal{Z}(\xi,\eta)$ *zigzagging* behaviours.

As an example consider scenarios ξ of Fig. 1 and η of Fig. 2. Fig. 3 shows $\xi \Cup \eta$ along with its stable table. Scenario ζ of Fig. 3 represents a set of behaviours in which the time distance between events a and b is exactly 2, and between events

[4] That is, a constraint cannot be removed without changing the semantics.

ξ:

$0 : a\ ;$
$1 : b\ \{\tau_{0,1} \geq 3\}\ ;$
$2 : c\ \{\tau_{0,2} \leq 7\}\ .$

	1	2
0	$(3, \infty)$	$(0, 7)$
1		$(0, \infty)$

	1	2
0	$(3, 7)$	$(3, 7)$
1		$(0, 4)$

Fig. 1. ξ, its initial table and its stable table

η:

$0 : a\ ;$
$1 : b\ ;$
$2 : c\ \{\tau_{1,2} \geq 4\}\ .$

	1	2
0	$(0, \infty)$	$(4, \infty)$
1		$(4, \infty)$

Fig. 2. η and its stable table

$\xi \Cup \eta$:

$0 : a\ ;$
$1 : b\ ;$
$2 : c\ \{\tau_{0,2} \geq 3\}\ .$

	1	2
0	$(0, \infty)$	$(3, \infty)$
1		$(0, \infty)$

ζ:

$0 : a\ ;$
$1 : b\ \{\tau_{0,1} \geq 2, \tau_{0,1} \leq 2\};$
$2 : c\ \{\tau_{1,2} \geq 3, \tau_{1,2} \leq 3\}\ .$

	1	2
0	(2,2)	(5,5)
1		(3,3)

Fig. 3. The combination of ξ and η of figures 1 and 2, and ζ with its stable table

ξ:

$0 : a\ ;$
$1 : b\ ;$
$2 : c\ \{\tau_{0,2} \leq 3\}\ ;$
$3 : d\ \{\tau_{0,3} \leq 5\}.$

	1	2	3
0	(0,3)	(0,3)	(0,5)
1		(0,3)	(0,5)
2			(0,5)

η:

$0 : a\ ;$
$1 : b\ ;$
$2 : c\ ;$
$3 : d\ \{\tau_{0,3} \geq 2, \tau_{1,3} \leq 5\}.$

	1	2	3
0	$(0,\infty)$	$(0,\infty)$	$(2,\infty)$
1		(0,5)	(0,5)
2			(0,5)

$\xi \cup \eta$:

$0 : a\ ;$
$1 : b\ ;$
$2 : c\ ;$
$3 : d\ \{\tau_{1,3} \leq 5\}.$

	1	2	3
0	$(0,\infty)$	$(0,\infty)$	$(0,\infty)$
1		(0,5)	(0,5)
2			(0,5)

Fig. 4. Two scenarios and their union

b and c is exactly 3 units of time. There is no behaviour in the semantics of ζ that is allowed by either ξ or η, yet $[\![\zeta]\!] \subset [\![\xi \Cup \eta]\!]$. That is, all behaviours in $[\![\zeta]\!]$ belong to $\mathcal{Z}(\xi, \eta)$. *This indicates that the union of the sets of behaviours allowed by ξ and η cannot be represented by a single scenario.*

Given two consistent scenarios ξ and η of length n with the same sequence of events, $\mathcal{E}$, such that $\xi \Cup \eta$ is defined, behaviour $\mathcal{B}^z$ with the sequence of events $\mathcal{E}$ belongs to $\mathcal{Z}(\xi, \eta)$ iff (1) For every $0 \leq i < j < n$, $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\xi \Cup \eta}$, (2) There exist $0 \leq i < j < n$ and $0 \leq k < l < n$ such that (a) $i \neq k$ or $j \neq l$, (b) $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$, $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\eta}$, and (c) $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$, $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$.

There is a sufficient condition for the non-existence of zigzagging behaviours [13]: if $\mathcal{Z}(\xi, \eta) \neq \emptyset$ then ξ and η each has at least one explicit constraint, such that the constraints are not between the same events i and j in both ξ and η.

If $\mathcal{Z}(\xi, \eta) = \emptyset$, $\xi \Cup \eta$ becomes their *union*, denoted by $\xi \cup \eta$. Then, scenario $\xi \cup \eta$ captures the union of the sets of behaviours allowed by ξ, η or both: $[\![\xi \cup \eta]\!] = [\![\xi]\!] \cup [\![\eta]\!]$. Fig. 4 shows two scenarios and their union.

3 Construction of zigzagging behaviours

In this section we develop a method for constructing zigzagging behaviours, along with the necessary definitions and theoretical foundations.

Definition 1 is a generalization of our previously-introduced notion of compatibility of a timed sequence with a stable distance table [11].

Definition 1. *Let ξ be a scenario of size n and $\mathcal{D}_s^\xi$ be its stable distance table. Let $0 \leq k < n$ and let $\mathcal{J} = \{b_0, b_1, \ldots, b_k\} \subseteq \{i \in \mathbb{N} \mid 0 \leq i < n\}$, where $b_0 < b_1 < \cdots < b_k$. The sequence of real numbers $S = t_{b_0} t_{b_1} \ldots t_{b_k}$ is* compatible *with $\mathcal{D}_s^\xi$ iff the following three conditions hold:*

1. $b_0 > 0 \Rightarrow t_{b_0} \geq m_{0b_0}$,
2. $\forall i, l \in \mathcal{J}\ (i < l \Rightarrow t_i \leq t_l)$ *and*
3. $\forall i, l \in \mathcal{J}\ (i < l \Rightarrow t_{il} \in I_{il}^\xi)$.

If $k = 0$ then the sequence consists of one item. Conditions 2 and 3 are then trivially true. An empty sequence is not compatible with any stable table.

The sequence $t_1 t_2$, where $t_1 = 3$ and $t_2 = 4$, is compatible with the stable table for scenario ξ of Fig. 4: $t_1 \geq m_{01} = 0$, $t_1 \leq t_2$ and $t_{12} = t_2 - t_1 = 1 \in I_{12}^\xi$.

Observation 1 *Let ξ be a scenario of size n, let $\mathcal{D}_s^\xi$ be its stable distance table and let $e_o e_1 .. e_{n-1}$ be the events of ξ. The sequence $S = t_0 t_1 \ldots t_{n-1}$ is compatible with $\mathcal{D}_s^\xi$ iff $(e_o, t_o)(e_1, t_1) \ldots (e_{n-1}, t_{n-1}) \in [\![\xi]\!]$.*

Intuitively, a sequence that is compatible with a stable table of a scenario is the sequence of time values shared by the members of a set of behaviours allowed by the scenario. If the sequence has n elements, then the set is a singleton, i.e., the sequence represents one "complete" behaviour. If the sequence has less than n elements, it can be thought of as a "partial" behaviour. Such a behaviour can always be extended to a complete behaviour, as shown in Theorem 1.

Theorem 1. *Let ξ be a scenario of size n and $\mathcal{D}_s^\xi$ be its stable distance table. Let $\mathcal{J} = \{b_0, b_1, \ldots, b_k\} \subseteq \{i \in \mathbb{N} \mid 0 \leq i < n\}$, $k \geq 0$, $b_0 < b_1 < \cdots < b_k$ and let $S = t_{b_0} t_{b_1} \ldots t_{b_k}$ be compatible with $\mathcal{D}_s^\xi$. Then, for any $r \in \{i \in \mathbb{N} \mid 0 \leq i < n\} \setminus \mathcal{J}$, there exists a real number t_r such that*

1. *if $b_k < r < n$, then $t_{b_0} \ldots t_{b_k} \ldots t_r$ is compatible with $\mathcal{D}_s^\xi$;*
2. *if $b_0 < r < b_k$, then $t_{b_0} \ldots t_r \ldots t_{b_k}$ is compatible with $\mathcal{D}_s^\xi$;*
3. *if $0 \leq r < b_0$, then $t_r \ldots t_{b_0} \ldots t_{b_k}$ is compatible with $\mathcal{D}_s^\xi$.*

Proof. (Fig. 5 illustrates the three cases of the theorem.) We show the proof for Case 1. The complete proof is presented in Appendix A.

Case 1: If $k = 0$, then let $i = b_0$ and $b_0 < r$. We must show that there exists a real number t_r such that $t_i \leq t_r$ and $m_{ir} \leq t_{ir} \leq M_{ir}$. But because $m_{ir} \leq M_{ir}$, it is always possible to find a t_{ir} that satisfies the inequations.

Let $k > 0$, $b_k < r$ and let $i = b_p$ and $j = b_q$, where $0 \leq p < q \leq k$. We must show that there exists a real number t_r such that $t_i \leq t_j \leq t_r$ and

$$m_{jr} \leq t_{jr} \leq M_{jr} \quad (2) \qquad m_{ir} \leq t_{ir} \leq M_{ir} \quad (3)$$

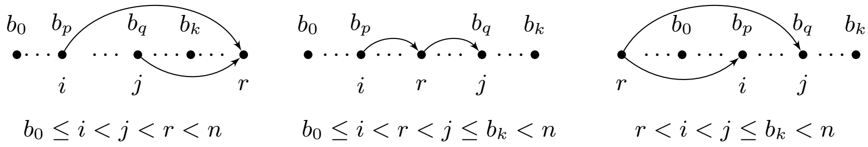

Fig. 5. Three cases of Theorem 1

Inequation (6) is equivalent to $m_{jr} \leq t_{ir} - t_{ij} \leq M_{jr}$, which is equivalent to

$$t_{ij} + m_{jr} \leq t_{ir} \leq t_{ij} + M_{jr} \quad (4)$$

Clearly, $t_{ij} + m_{jr} \leq t_{ij} + M_{jr}$, because $m_{jr} \leq M_{jr}$. Moreover, $0 \leq t_{ij} + m_{jr}$, because neither of the terms is negative. So it is possible to find a t_{ir} that satisfies (8), therefore (6) can be satisfied.

Inequation (7) is equivalent to $m_{ir} \leq t_{ij} + t_{jr} \leq M_{ir}$ which is equivalent to

$$m_{ir} - t_{ij} \leq t_{jr} \leq M_{ir} - t_{ij} \quad (5)$$

Obviously, $m_{ir} - t_{ij} \leq M_{ir} - t_{ij}$, because $m_{ir} \leq M_{ir}$.

It is easy to show that $0 \leq M_{ir} - t_{ij}$:

$\mathcal{D}_s^\xi$ is stable, so $M_{ij} + m_{jr} \leq M_{ir}$, therefore $m_{jr} \leq M_{ir} - M_{ij}$. Since $t_{ij} \leq M_{ij}$, $M_{ir} - M_{ij} \leq M_{ir} - t_{ij}$. But $0 \leq m_{jr}$, therefore $0 \leq M_{ir} - t_{ij}$.

So it is possible to find a t_{jr} that satisfies (9), therefore (7) can be satisfied.

Next, we must show that the same value of t_r can simultaneously satisfy both (6) and (7), or—equivalently—both (6) and (9). We do so by showing that $m_{jr} \leq M_{ir} - t_{ij}$ and $m_{ir} - t_{ij} \leq M_{jr}$.

$\mathcal{D}_s^\xi$ is stable, so $M_{ij} + m_{jr} \leq M_{ir}$, therefore $m_{jr} \leq M_{ir} - M_{ij} \leq M_{ir} - t_{ij}$ (because $t_{ij} \leq M_{ij}$).

Similarly, $m_{ir} \leq m_{ij} + M_{jr}$, therefore $m_{ir} - m_{ij} \leq M_{jr}$. But $m_{ir} - t_{ij} \leq m_{ir} - m_{ij}$ (because $m_{ij} \leq t_{ij}$), so $m_{ir} - t_{ij} \leq M_{jr}$.

Intuitively, this means that none of the lower bounds on t_r imposed by (6) and (7) exceeds any of the upper bounds imposed by these inequations, for any choice of i and j that satisfies the assumptions. □

For example, the sequence $t_1 t_2$, where $t_1 = 3$ and $t_2 = 4$, which is compatible with the stable table for scenario ξ of Fig. 4, can be extended to $t_0 t_1 t_2$, where $t_0 = 1$, or to $t_1 t_2 t_3$, where $t_3 = 6$, or to $t_0 t_1 t_2 t_3$. All of the extended sequences are compatible with the stable table for ξ. The last one corresponds to a complete behaviour allowed by ξ: $(a, 1)(b, 3)(c, 4)(d, 6) \in [\![\xi]\!]$.

Definition 2. *Let $\mathcal{D}_s^\xi$ be the stable distance table for scenario ξ of length n. Let $0 \leq i < j < n$, $m_{ij} \neq M_{ij}$ and $v \in I_{ij}^\xi$. The action of setting both l_{ij} and h_{ij} in $\mathcal{D}_s^\xi$ to v is called* collapsing I_{ij}^ξ to v. *Then I_{ij}^ξ is a* collapsed *interval.*

When an interval is collapsed to a value, the distance table may cease to be stable. Moreover, both the table and the interval are now associated with a different scenario, namely ξ augmented with the additional constraint $t_{ij} = v$.

Observation 2 *Let $\mathcal{D}_s^\xi$ be the stable distance table for scenario ξ of length n. After collapsing I_{ij}^ξ to a value in I_{ij}^ξ, $\mathcal{D}_s^\xi$ can be successfully stabilised (i.e., stabilisation does not make the table invalid).*

Proof. Let $v \in I_{ij}^\xi$. Collapsing I_{ij}^ξ to v is equivalent to imposing the additional constraints $t_{ij} \geq v, t_{ij} \leq v$. Consider the sequence $s = t_i t_j$, such that $t_i \geq m_{0i}^\xi$ and $t_j - t_i = v$. Clearly, s is compatible with $\mathcal{D}_s^\xi$. By Theorem 1, s can be extended to a complete behaviour $\mathcal{B} \in [\![\xi]\!]$. Behaviour $\mathcal{B}$ corresponds to a scenario, say η, such that for $0 \leq r < s < n$, if $t_{rs}^{\mathcal{B}} = u$, then $\mathcal{D}^\eta[r,s] = (u,u)$. That is, $m_{rs}^\eta = M_{rs}^\eta = u$. We show that $\mathcal{D}^\eta$ is stable:

For every $0 \leq i < j < k < n$: $m_{ij}^\eta = M_{ij}^\eta = t_{ij}^{\mathcal{B}}$, $m_{jk}^\eta = M_{jk}^\eta = t_{jk}^{\mathcal{B}}$, $m_{ik}^\eta = M_{ik}^\eta = t_{ik}^{\mathcal{B}}$. Inequation (1) becomes

$$t_{ij} + t_{jk} \leq t_{ik} \leq \left\{ \begin{array}{c} t_{ij} + t_{jk} \\ t_{ij} + t_{jk} \end{array} \right\} \leq t_{ik} \leq t_{ij} + t_{jk}$$

By definition, $t_{ij} + t_{jk} = (t_j - t_i) + (t_k - t_j) = t_k - t_i = t_{ik}$. So $\mathcal{D}^\eta$ satisfies all the inequations. That is, $\mathcal{D}^\eta$ is stable, which means that, after collapsing I_{ij}^ξ, stabilization could have not made $\mathcal{D}_s^\xi$ invalid, since every interval in the stabilized table must contain the corresponding interval of $\mathcal{D}^\eta$. □

It will sometimes be convenient to not annotate the symbols with the name of the scenario, i.e., to write I_{ij} and $\mathcal{D}$, instead of I_{ij}^ξ and $\mathcal{D}^\xi$.

Observation 3 *Let $\mathcal{D}$ be a stable distance table for a scenario of length n, $0 \leq i < j < n$, $0 \leq k < l < n$ and $i \neq k \vee j \neq l$. If I_{ij} and I_{kl} are collapsed intervals, then it is possible to construct a sequence $S = t_{b_0} t_{b_1} t_{b_2} t_{b_3}$ such that $\{b_0, b_1, b_2, b_3\} \subseteq \{i, j, k, l\}$ and S is compatible with $\mathcal{D}$.*

Proof. The proof is presented in Appendix B.

4 *z_pairs*

In our earlier work [13] we proved that if $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ are two optimized scenarios of length n, such that $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then there are constraints $\alpha = \tau_{i,j} \sim a \in \mathcal{C}_1$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}_2$ (for some $a, b \in \mathbb{Q}$, $0 \leq i < j < n$, $0 \leq k < l < n$), such that $\alpha \notin \mathcal{C}_2$, $\beta \notin \mathcal{C}_1$ and ($i \neq k$ or $j \neq l$). That is, the absence of such a pair of explicit constraints implies that there is no zigzagging.

It turns out that in the presence of zigzagging behaviours the constraints of ξ and η are involved in some interesting relations, which depend on the positions of i, j, k and l with respect to each other. This will allow us to formulate a stronger condition: zigzagging is present if and only if the condition is satisfied (Theorem 2 and Theorem 3).

Definition 3. *Let ξ and η be two scenarios of length n. If behaviour $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$ is such that $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^\xi$, $t_{ij}^{\mathcal{B}^z} \in I_{ij}^\eta$, $t_{kl}^{\mathcal{B}^z} \in I_{kl}^\xi$, $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^\eta$ and $i \neq k \vee j \neq l$ ($0 \leq i < j < n, 0 \leq k < l < n$), then we say $\mathcal{B}^z$* zigzags through *ij and kl.*

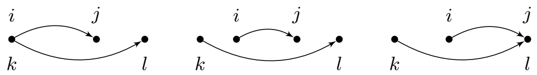

Fig. 6. case 1 of Observation 3 and of Definition 4

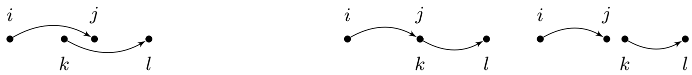

Fig. 7. cases 2, 3 of Observation 3 and of Definition 4

As an example consider ξ and η of figures 1 and 2. Behaviour $\mathcal{B}^z = (a,0)(b,2)(c,5)$ zigzags through 01 and 12: $t_{01}^{\mathcal{B}^z} \notin I_{01}^{\xi}$, $t_{01}^{\mathcal{B}^z} \in I_{01}^{\eta}$, $t_{12}^{\mathcal{B}^z} \notin I_{12}^{\eta}$ and $t_{12}^{\mathcal{B}^z} \in I_{12}^{\xi}$.

Definition 4. *Let ξ and η be two scenarios of length n with the same sequences of events, such that $\xi \not\subseteq \eta$, $\eta \not\subseteq \xi$ and $\xi \Cup \eta$ is defined. Let $\alpha = \tau_{i,j} \sim a \in \mathcal{C}(\mathcal{D}_s^{\xi})$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}(\mathcal{D}_s^{\eta})$ $(a, b \in \mathbb{Q})$, such that $i \neq k \vee j \neq l$, $\alpha \notin \mathcal{C}(\mathcal{D}_s^{\eta})$ and $\beta \notin \mathcal{C}(\mathcal{D}_s^{\xi})$. Constraints α and β form a z_pair if one of the following conditions holds (see the diagrams in figures 6 and 7):*

(1) $0 \le k \le i < j \le l < n$ *and*
- *(a)* $\alpha = \tau_{i,j} \ge a$, $\beta = \tau_{k,l} \ge b$, $m_{ij}^{\eta} < a$, $m_{kl}^{\xi} < b$, *or*
- *(b)* $\alpha = \tau_{i,j} \ge a$, $\beta = \tau_{k,l} \le b$, $m_{ij}^{\eta} < a$, $b < M_{kl}^{\xi}$, *and additionally* $M_{ki}^{\xi\Cup\eta} + a + M_{jl}^{\xi\Cup\eta} > b$, *or*
- *(c)* $\alpha = \tau_{i,j} \le a$, $\beta = \tau_{k,l} \ge b$, $a < M_{ij}^{\eta}$, $m_{kl}^{\xi} < b$, *and additionally* $m_{ki}^{\xi\Cup\eta} + a + m_{jl}^{\xi\Cup\eta} < b$, *or*
- *(d)* $\alpha = \tau_{i,j} \le a$, $\beta = \tau_{k,l} \le b$, $a < M_{ij}^{\eta}$, $b < M_{kl}^{\xi}$.

(2) $0 \le i < k < j < l < n$ *and*
- *(a)* $\alpha = \tau_{i,j} \ge a$, $\beta = \tau_{k,l} \ge b$, $m_{ij}^{\eta} < a$, $m_{kl}^{\xi} < b$, *and additionally* $m_{il}^{\xi\Cup\eta} - a < b - m_{kj}^{\xi\Cup\eta}$, *or*
- *(b)* $\alpha = \tau_{i,j} \ge a$, $\beta = \tau_{k,l} \le b$, $m_{ij}^{\eta} < a$, $b < M_{kl}^{\xi}$, *and additionally* $a + M_{jl}^{\xi\Cup\eta} > m_{ik}^{\xi\Cup\eta} + b$, *or*
- *(c)* $\alpha = \tau_{i,j} \le a$, $\beta = \tau_{k,l} \ge b$, $a < M_{ij}^{\eta}$, $m_{kl}^{\xi} < b$, *and additionally* $a + m_{jl}^{\xi\Cup\eta} < M_{ik}^{\xi\Cup\eta} + b$, *or*
- *(d)* $\alpha = \tau_{i,j} \le a$, $\beta = \tau_{k,l} \le b$, $a < M_{ij}^{\eta}$, $b < M_{kl}^{\xi}$, *and additionally* $M_{il}^{\xi\Cup\eta} - a > b - M_{kj}^{\xi\Cup\eta}$.

(3) $0 \le i < j \le k < l < n$ *and*
- *(a)* $\alpha = \tau_{i,j} \ge a$, $\beta = \tau_{k,l} \ge b$, $m_{ij}^{\eta} < a$, $m_{kl}^{\xi} < b$, *or*
- *(b)* $\alpha = \tau_{i,j} \ge a$, $\beta = \tau_{k,l} \le b$, $m_{ij}^{\eta} < a$, $b < M_{kl}^{\xi}$, *or*
- *(c)* $\alpha = \tau_{i,j} \le a$, $\beta = \tau_{k,l} \ge b$, $a < M_{ij}^{\eta}$, $m_{kl}^{\xi} < b$, *or*
- *(d)* $\alpha = \tau_{i,j} \le a$, $\beta = \tau_{k,l} \le b$, $a < M_{ij}^{\eta}$, $b < M_{kl}^{\xi}$.

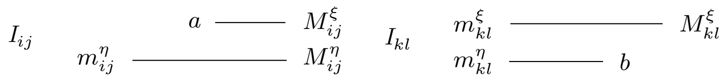

Fig. 8. Definition 4 cases (1)(b), (2)(b) and (3)(b): $\tau_{i,j} \geq a$, $\tau_{k,l} \leq b$, $m^{\eta}_{ij} < a$, $b < M^{\xi}_{kl}$

In each case there are four other subcases which can be obtained by interchanging ξ and η.

The three cases of Definition 4 cover all[5] the possibilities of the positions of i, j, k and l with respect to each other. The "main" subcases of each case cover all the possible combinations of the forms of α and β, and the relations between the minima and maxima of distances between events i and j and between events k and l in ξ and η. Intuitively, the conditions capture all the possibilities for $I^{\eta}_{ij} \setminus I^{\xi}_{ij} \neq \emptyset$ and $I^{\xi}_{kl} \setminus I^{\eta}_{kl} \neq \emptyset$, to guarantee "there is room" for behaviours to zigzag through ij and kl. But these are only half of the possibilities: The cases for $I^{\xi}_{ij} \setminus I^{\eta}_{ij} \neq \emptyset$ and $I^{\eta}_{kl} \setminus I^{\xi}_{kl} \neq \emptyset$ are obtained by exchanging ξ and η.

Fig. 8 is a schematic illustration of the main conditions of cases (1)(b), (2)(b) and (3)(b): $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \leq b$, $m^{\eta}_{ij} < a$, $b < M^{\xi}_{kl}$. Because of constraint α, $m^{\xi}_{ij} = a$ and because of constraint β, $M^{\eta}_{kl} = b$. Therefore, $I^{\eta}_{ij} \setminus I^{\xi}_{ij} = \{u \in \mathbb{Q}_{\geq 0} \mid m^{\eta}_{kl} \leq u < a\} \neq \emptyset$ and $I^{\xi}_{kl} \setminus I^{\eta}_{kl} = \{v \in \mathbb{Q}_{\geq 0} \mid b < v < M^{\xi}_{kl}\} \neq \emptyset$.

The *additional* conditions specify certain relations that must hold between various minima and maxima in ξ and η for there to be zigzagging behaviours.

It is worth mentioning that if α and β form a *z_pair*, then neither of them can be a default constraint.

In the rest of the paper we are going to prove theorems that will show the following relations between the existence of a *z_pair* and the existence of the union of two scenarios:

- no *z_pair* $\Rightarrow$ no zigzagging $\Rightarrow$ union (Theorem 2);
- *z_pair* $\Rightarrow$ zigzagging $\Rightarrow$ no union (Theorem 3).

5 A sufficient condition for the existence of union

Theorem 2. *Let ξ and η be two scenarios of length n with the same sequences of events, such that $\xi \not\subseteq \eta$ and $\eta \not\subseteq \xi$. If $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$ zigzags through ij and kl, then there exist $\alpha = \tau_{i,j} \sim a \in \mathcal{C}(\mathcal{D}^{\xi}_s)$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}(\mathcal{D}^{\eta}_s)$ $(a, b \in \mathbb{Q})$ such that α and β form a z_pair.*

Proof. We show the proof for case (1) of Definition 4: $0 \leq k \leq i < j \leq l < n$. The complete proof is presented in Appendix C.

[5] Strictly speaking, there are three other cases where ij and kl are interchanged, but we need not consider them in the definition.

Let $\mathcal{B}^z$ be a behaviour in $\mathcal{Z}(\xi, \eta)$ that zigzags through ij and kl. There are two cases to consider:

(1) $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$, $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\eta}$, $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$ and $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$.
Since $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$, one of the following two cases must hold:
  (a) $t_{ij}^{\mathcal{B}^z} < m_{ij}^{\xi}$. Since $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\eta}$, we must have $m_{ij}^{\eta} \leq t_{ij}^{\mathcal{B}^z}$. It follows that $m_{ij}^{\eta} < m_{ij}^{\xi}$. Therefore $m_{ij}^{\xi} > 0$: there exists a non-default constraint $\alpha = \tau_{i,j} \geq a$ in $\mathcal{C}(\mathcal{D}_s^{\xi})$, where $a = m_{ij}^{\xi}$.
  Since $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$, one of the following two cases must hold:
  – $t_{kl}^{\mathcal{B}^z} < m_{kl}^{\eta}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$, we must have $m_{kl}^{\xi} \leq t_{kl}^{\mathcal{B}^z}$. It follows that $m_{kl}^{\xi} < m_{kl}^{\eta}$. Therefore $m_{kl}^{\eta} > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}_s^{\eta})$, where $b = m_{kl}^{\eta}$.
  So α and β form a *z_pair* (case (1)(a) of Definition 4).
  – $M_{kl}^{\eta} < t_{kl}^{\mathcal{B}^z}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$, we must have $t_{kl}^{\mathcal{B}^z} \leq M_{kl}^{\xi}$. It follows that $M_{kl}^{\eta} < M_{kl}^{\xi}$. Therefore $M_{kl}^{\eta} < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}_s^{\eta})$, where $b = M_{kl}^{\eta}$.
  The constraints α and β fit case (1)(b) of Definition 4. We must show $M_{ki}^{\xi \Cup \eta} + a + M_{jl}^{\xi \Cup \eta} > b$.
  Assume $M_{ki}^{\xi \Cup \eta} + a + M_{jl}^{\xi \Cup \eta} \leq b$.
  By definition, $t_{kl}^{\mathcal{B}^z} = t_{ki}^{\mathcal{B}^z} + t_{ij}^{\mathcal{B}^z} + t_{jl}^{\mathcal{B}^z}$. Because $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$, we have $t_{ki}^{\mathcal{B}^z} \leq M_{ki}^{\xi \Cup \eta}$ and $t_{jl}^{\mathcal{B}^z} \leq M_{jl}^{\xi \Cup \eta}$. Therefore, $t_{kl}^{\mathcal{B}^z} \leq M_{ki}^{\xi \Cup \eta} + t_{ij}^{\mathcal{B}^z} + M_{jl}^{\xi \Cup \eta}$. Since $t_{ij}^{\mathcal{B}^z} < m_{ij}^{\xi}$ and $m_{ij}^{\xi} = a$, we have $t_{kl}^{\mathcal{B}^z} < M_{ki}^{\xi \Cup \eta} + a + M_{jl}^{\xi \Cup \eta}$. By the assumption, $M_{ki}^{\xi \Cup \eta} + a + M_{jl}^{\xi \Cup \eta} \leq b$. So it follows that $t_{kl}^{\mathcal{B}^z} < b$. But $M_{kl}^{\eta} < t_{kl}^{\mathcal{B}^z}$, so $M_{kl}^{\eta} < b$: a contradiction.
  So α and β form a *z_pair* (case (1)(b) of Definition 4).
  (b) $M_{ij}^{\xi} < t_{ij}^{\mathcal{B}^z}$. Since $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\eta}$, we must have $t_{ij}^{\mathcal{B}^z} \leq M_{ij}^{\eta}$. It follows that $M_{ij}^{\xi} < M_{ij}^{\eta}$. Therefore $M_{ij}^{\xi} < \infty$: there exists a non-default constraint $\alpha = \tau_{i,j} \leq a$ in $\mathcal{C}(\mathcal{D}_s^{\xi})$, where $a = M_{ij}^{\xi}$.
  Since $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$, one of the following two cases must hold:
  – $t_{kl}^{\mathcal{B}^z} < m_{kl}^{\eta}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$, we must have $m_{kl}^{\xi} \leq t_{kl}^{\mathcal{B}^z}$. It follows that $m_{kl}^{\xi} < m_{kl}^{\eta}$. Therefore $m_{kl}^{\eta} > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}_s^{\eta})$, where $b = m_{kl}^{\eta}$.
  The constraints α and β fit case (1)(c) of Definition 4. We must show $m_{ki}^{\xi \Cup \eta} + a + m_{jl}^{\xi \Cup \eta} < b$.
  Assume $m_{ki}^{\xi \Cup \eta} + a + m_{jl}^{\xi \Cup \eta} \geq b$.
  By definition, $t_{kl}^{\mathcal{B}^z} = t_{ki}^{\mathcal{B}^z} + t_{ij}^{\mathcal{B}^z} + t_{jl}^{\mathcal{B}^z}$. Because $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$, we have $t_{ki}^{\mathcal{B}^z} \geq m_{ki}^{\xi \Cup \eta}$ and $t_{jl}^{\mathcal{B}^z} \geq m_{jl}^{\xi \Cup \eta}$. Therefore, $t_{kl}^{\mathcal{B}^z} \geq m_{ki}^{\xi \Cup \eta} + t_{ij}^{\mathcal{B}^z} + m_{jl}^{\xi \Cup \eta}$. Since $t_{ij}^{\mathcal{B}^z} > M_{ij}^{\xi}$ and $M_{ij}^{\xi} = a$, we have $t_{kl}^{\mathcal{B}^z} > m_{ki}^{\xi \Cup \eta} + a + m_{jl}^{\xi \Cup \eta}$. By the assumption, $m_{ki}^{\xi \Cup \eta} + a + m_{jl}^{\xi \Cup \eta} \geq b$. So it follows that $t_{kl}^{\mathcal{B}^z} > b$. But $m_{kl}^{\eta} > t_{kl}^{\mathcal{B}^z}$, so $m_{kl}^{\eta} > b$: a contradiction.
  So α and β form a *z_pair* (case (1)(c) of Definition 4).

– $M^\eta_{kl} < t^{\mathcal{B}^z}_{kl}$. Since $t^{\mathcal{B}^z}_{kl} \in I^\xi_{kl}$, we must have $t^{\mathcal{B}^z}_{kl} \leq M^\xi_{kl}$. It follows that $M^\eta_{kl} < M^\xi_{kl}$. Therefore $M^\eta_{kl} < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}^\eta_s)$, where $b = M^\eta_{kl}$.
So α and β form a *z_pair* (case (1)(d) of Definition 4).

(2) $t^{\mathcal{B}^z}_{ij} \in I^\xi_{ij}$, $t^{\mathcal{B}^z}_{ij} \notin I^\eta_{ij}$, $t^{\mathcal{B}^z}_{kl} \in I^\eta_{kl}$ and $t^{\mathcal{B}^z}_{kl} \notin I^\xi_{kl}$.
The proof can be obtained by exchanging ξ and η in case (1).

□

As an example consider ξ and η of figures 1 and 2 along with their stable distance tables once more. As we mentioned before, $\mathcal{Z}(\xi, \eta) \neq \emptyset$ and there is a behaviour that zigzags through 01 and 12. For example, $\mathcal{B}^z = (a, 0)(b, 1)(c, 4)$ is such a behaviour: $t^{\mathcal{B}^z}_{01} = 1 - 0 \notin I^\xi_{01}$ and $t^{\mathcal{B}^z}_{12} = 4 - 1 \notin I^\eta_{12}$. According to Theorem 2 at least one *z_pair* must exist between ξ and η. Indeed, constraints $\alpha = \tau_{0,1} \geq 3$ in ξ and $\beta = \tau_{1,2} \geq 4$ in η satisfy the requirements of Definition 4 (case 3.a), and therefore, form a *z_pair*: $i = 0, j = k = 1, l = 2, i < j = k < l$, $a = 3, b = 4$, $m^\eta_{01} = 0 < a = 3$ and $m^\xi_{12} = 0 < b = 4$.

As another example consider scenarios ξ and η of Fig. 4. Constraints $\tau_{0,2} \leq 3$ in ξ and $\tau_{0,3} \geq 2$ in η do not form a *z_pair*: $i = k = 0, j = 2, l = 3$, and $a = 3 < M^\eta_{02} = \infty$, $m^\xi_{03} = 0 < b = 2$ (case 1.c), but $m^{\xi \Cup \eta}_{02} + a + m^{\xi \Cup \eta}_{23} = 0 + 3 + 0 \not< b = 2$. Constraints $\tau_{0,3} \leq 5$ in ξ and $\tau_{1,3} \leq 5$ in η do not form a *z_pair*: $I^\xi_{13} = I^\eta_{13}$. Constraints $\tau_{0,2} \leq 3$ in ξ and $\tau_{1,3} \leq 5$ in η do not form a *z_pair* for the same reason. Of course, constraints $\tau_{0,3} \leq 5$ in ξ and $\tau_{0,3} \geq 2$ in η do not form a *z_pair*: $i = k = 0$, $j = l = 3$, so $i = k \wedge j = l$.

In fact, none of the constraints of ξ and η satisfy the conditions of any of the cases of Definition 4: there is no *z_pair* between constraints of ξ and η. This is in accordance with Theorem 2: as we mentioned before, $\mathcal{Z}(\xi, \eta) = \emptyset$.

The consequence of Theorem 2 is that if there is no *z_pair* between scenarios ξ and η, then $\mathcal{Z}(\xi, \eta) = \emptyset$, therefore $\xi \cup \eta$ exists. That is, the non-existence of *z_pair*s between two scenarios is a sufficient condition for the existence of the union between the two (provided $\xi \Cup \eta$ is defined).

In our earlier work [13] we identified another such condition: if scenarios ξ and η are optimized and $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then ξ and η must each have at least one explicit constraint that the other does not have, such that the two constraints are not betwen the same events in both ξ and η. The absence of such a pair of explicit constraints is also a sufficient condition for the existence of $\xi \cup \eta$, but the pair need not be a *z_pair*.

The existence of a *z_pair* turns out to be not only a necessary, but also a sufficient condition for the existence of zigzagging behaviours (see Theorem 3).

Consider scenarios ξ of Fig. 9 and η of Fig. 10: $\mathcal{Z}(\xi, \eta) \neq \emptyset$. Indeed, constraints $\tau_{0,2} \leq 2$ of ξ and $\tau_{1,3} \geq 4$ of η are two different explicit constraints between different events, events 0 and 2, and events 1 and 3, respectively. However, the two constraints do not form a *z_pair*: their form fits case (2)(c) of Definition 4, however $a = 2 \not< M^\eta_{02} = 2$. But, of course, there must exist at least one *z_pair*. Constraint $\tau_{0,2} \geq 1$ of ξ, which is an implied constraint, and constraint $\tau_{1,3} \geq 4$ of η form a *z_pair*.

$0 : a$;
$1 : b\ \{\tau_{0,1} \geq 1\}$;
$2 : c\ \{\tau_{0,2} \leq 2\}$;
$3 : d$.

	1	2	3
0	$(1,2)$	$(1,2)$	$(1,\infty)$
1		$(0,1)$	$(0,\infty)$
2			$(0,\infty)$

Fig. 9. ξ and its stable table

$0 : a$;
$1 : b\ \{\tau_{0,1} \leq 1\}$;
$2 : c\ \{\tau_{1,2} \leq 1\}$;
$3 : d\ \{\tau_{1,3} \geq 4\}$.

	1	2	3
0	$(0,1)$	$(0,2)$	$(4,\infty)$
1		$(0,1)$	$(4,\infty)$
2			$(3,\infty)$

Fig. 10. η and its stable table

6 A sufficient condition for the non-existence of union

Our goal is to show that the existence of a *z_pair* between two scenarios ξ and η is a sufficient condition for the non-existence of $\xi \cup \eta$, because of the existence of zigzagging behaviours between the two scenarios. More precisely, if there is a *z_pair* between ξ and η, then there exist $0 \leq i < j < n$, $0 \leq k < l < n$, $u \in I_{ij}^{\xi \Cup \eta}$, $v \in I_{kl}^{\xi \Cup \eta}$ and behaviour $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$, such that $t_{ij}^{\mathcal{B}^z} = u \in I_{ij}^{\xi} \setminus I_{ij}^{\eta}$ and $t_{kl}^{\mathcal{B}^z} = v \in I_{kl}^{\eta} \setminus I_{kl}^{\xi}$.

Observation 4 *Let $\mathcal{D}$ be a stable distance table for a scenario of length n and let $0 \leq i \leq j \leq k < n$ be integers.*

Let t_{ij} satisfy $m_{ij} \leq t_{ij} \leq M_{ij}$. Then, after replacing m_{ij} and M_{ij} with t_{ij} and stabilising the table, if m_{ik} changes, it will increase to at most $t_{ij} + m_{jk}$.

Proof. The proof is presented in Appendix D.

Observation 5 *Let $\mathcal{D}$ be a stable distance table for a scenario of length n and let $0 \leq i \leq j \leq k < n$ be integers. Let t_{ij} satisfy $m_{ij} \leq t_{ij} \leq M_{ij}$. Then, after replacing m_{ij} and M_{ij} with t_{ij}, if the inequation $t_{ij} + m_{jk} \leq m_{ik}$ holds, then during stabilisation m_{ik} will not change.*

Proof. Assume m_{ik} will change during stabilisation. In that case, $m_{ik} < \boldsymbol{m}_{ik}$, and By Observation 4, $\boldsymbol{m}_{ik} \leq t_{ij} + m_{jk}$.

By the assumption, the inequation $t_{ij} + m_{jk} \leq m_{ik}$ holds. It follows that $\boldsymbol{m}_{ik} \leq m_{ik}$: a contradiction with $m_{ik} < \boldsymbol{m}_{ik}$. □

Theorem 3. *Let ξ and η be two scenarios of length n with the same sequences of events, such that $\xi \not\subseteq \eta$, $\eta \not\subseteq \xi$ and $\xi \Cup \eta$ is defined. If there are $\alpha = \tau_{i,j} \sim a \in \mathcal{C}(\mathcal{D}_s^{\xi})$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}(\mathcal{D}_s^{\eta})$ $(a, b \in \mathbb{Q})$, such that α and β form a z_pair, then there is a $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$, such that $\mathcal{B}^z$ zigzags through ij and kl.*

Proof. Just as in Definition 4, there are three cases, determined by the relative positions of i, j, k and l. Each of these cases has four subcases, determined by the relations between the minima and maxima of the relevant intervals. We show the proof for case (1)(a). The complete proof is presented in Appendix E.

Case (1): $0 \leq k \leq i < j \leq l < n$ $(i \neq k \vee j \neq l)$.

We show the proof for the case $0 \leq k < i < j < l < n$. The proofs for $k = i$ or $j = l$ are special cases of the proof shown below.

We must show that there is a behaviour $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$ such that $\mathcal{B}^z$ zigzags through ij and kl. Because $k < i < j < l$, we must have $t_{ij}^{\mathcal{B}^z} \leq t_{kl}^{\mathcal{B}^z}$.

(a) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \geq b$, $m^{\eta}_{ij} < a$, $m^{\xi}_{kl} < b$

Because of constraint α, $m^{\xi}_{ij} = a$ and because of constraint β, $m^{\eta}_{kl} = b$. $m^{\xi \Cup \eta}_{ij} = \min(m^{\xi}_{ij}, m^{\eta}_{ij}) = m^{\eta}_{ij}$ and $m^{\xi \Cup \eta}_{kl} = \min(m^{\xi}_{kl}, m^{\eta}_{kl}) = m^{\xi}_{kl}$.
Both $I^{\xi \Cup \eta}_{ij} \backslash I^{\xi}_{ij} = \{u \in \mathbb{Q}_{>0} \mid m^{\eta}_{ij} \leq u < a\}$ and $I^{\xi \Cup \eta}_{kl} \backslash I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid m^{\xi}_{kl} \leq w < b\}$ are non-empty. So it is possible to pick a value $t_{ij} = a - \delta$ ($\delta \in \mathbb{Q}_{>0}$) in the interval $I^{\xi \Cup \eta}_{ij} \setminus I^{\xi}_{ij}$.
Next we must show that after collapsing $I^{\xi \Cup \eta}_{ij}$ of $\mathcal{D}^{\xi \Cup \eta}_{s}$ to t_{ij} (i.e., setting $m^{\xi \Cup \eta}_{ij} = M^{\xi \Cup \eta}_{ij} = t_{ij}$ and stabilizing $\mathcal{D}^{\xi \Cup \eta}_{s}$), hence obtaining $\boldsymbol{D}^{\xi \Cup \eta}_{s}$, it is still possible to pick a value t_{kl} in the interval $\boldsymbol{I}^{\xi \Cup \eta}_{kl}$ of table $\boldsymbol{D}^{\xi \Cup \eta}_{s}$, such that t_{kl} belongs to I^{ξ}_{kl}, but does not belong to I^{η}_{kl}, and that $t_{ij} \leq t_{kl}$.
(We use bold font to distinguish items that are updated during stabilization.)
In $\mathcal{D}^{\xi \Cup \eta}_{s}$ we have $m^{\xi \Cup \eta}_{ki} + m^{\xi \Cup \eta}_{ij} \leq m^{\xi \Cup \eta}_{kj}$ and $m^{\xi \Cup \eta}_{kj} + m^{\xi \Cup \eta}_{jl} \leq m^{\xi \Cup \eta}_{kl}$.
We set $\boldsymbol{m}^{\xi \Cup \eta}_{ij} = \boldsymbol{M}^{\xi \Cup \eta}_{ij} = t_{ij}$ in interval $\boldsymbol{I}^{\xi \Cup \eta}_{ij}$ of $\mathcal{D}^{\xi \Cup \eta}_{s}$ stabilise it to obtain $\boldsymbol{D}^{\xi \Cup \eta}_{s}$. After this one of the following two things will happen:

(i) The inequation $m^{\xi \Cup \eta}_{ki} + t_{ij} \leq m^{\xi \Cup \eta}_{kj}$ holds, in which case, by Observation 5, $m^{\xi \Cup \eta}_{kj}$ will not change: $\boldsymbol{m}^{\xi \Cup \eta}_{kj} = m^{\xi \Cup \eta}_{kj}$. The inequation $m^{\xi \Cup \eta}_{kj} + m^{\xi \Cup \eta}_{jl} \leq m^{\xi \Cup \eta}_{kl}$ will then continue to hold. Therefore $m^{\xi \Cup \eta}_{kl}$ will not change: $\boldsymbol{m}^{\xi \Cup \eta}_{kl} = m^{\xi \Cup \eta}_{kl} = m^{\xi}_{kl}$. Hence, $\boldsymbol{I}^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid m^{\xi}_{kl} \leq w < b\}$.

(ii) The inequation $m^{\xi \Cup \eta}_{ki} + t_{ij} \leq m^{\xi \Cup \eta}_{kj}$ does not hold. In that case, to restore the inequation, $m^{\xi \Cup \eta}_{kj}$ will be set to $m^{\xi \Cup \eta}_{ki} + t_{ij}$. By Observation 4, $m^{\xi \Cup \eta}_{kj}$ cannot further increase, therefore $\boldsymbol{m}^{\xi \Cup \eta}_{kj} = m^{\xi \Cup \eta}_{ki} + a - \delta$. But since $\boldsymbol{m}^{\xi \Cup \eta}_{kj}$ has changed, the inequation $\boldsymbol{m}^{\xi \Cup \eta}_{kj} + m^{\xi \Cup \eta}_{jl} \leq m^{\xi \Cup \eta}_{kl}$ might not hold. There are two cases to consider:
   1. If the new value of $\boldsymbol{m}^{\xi \Cup \eta}_{kj}$ does not affect the satisfiability of the inequation $\boldsymbol{m}^{\xi \Cup \eta}_{kj} + m^{\xi \Cup \eta}_{jl} \leq m^{\xi \Cup \eta}_{kl}$, then $m^{\xi \Cup \eta}_{kl}$ will remain the same, so just as in case (i), $\boldsymbol{I}^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid m^{\xi}_{kl} \leq w < b\}$.
   2. If $\boldsymbol{m}^{\xi \Cup \eta}_{kj} + m^{\xi \Cup \eta}_{jl} \leq m^{\xi \Cup \eta}_{kl}$ is not satisfied, the inequation will be restored by setting $m^{\xi \Cup \eta}_{kl}$ to $\boldsymbol{m}^{\xi \Cup \eta}_{kj} + m^{\xi \Cup \eta}_{jl}$: $\boldsymbol{m}^{\xi \Cup \eta}_{kl} := \boldsymbol{m}^{\xi \Cup \eta}_{kj} + m^{\xi \Cup \eta}_{jl} = m^{\xi \Cup \eta}_{ki} + a - \delta + m^{\xi \Cup \eta}_{jl}$. Now the inequation $\boldsymbol{m}^{\xi \Cup \eta}_{kj} + m^{\xi \Cup \eta}_{jl} \leq \boldsymbol{m}^{\xi \Cup \eta}_{kl}$ holds, and by Observation 5, $\boldsymbol{m}^{\xi \Cup \eta}_{kl}$ cannot be further increased. Next, we show that $\boldsymbol{m}^{\xi \Cup \eta}_{kl} < b$. For that, we show that $m^{\xi \Cup \eta}_{ki} + a + m^{\xi \Cup \eta}_{jl} < b$.
   Assume $b \leq m^{\xi \Cup \eta}_{ki} + a + m^{\xi \Cup \eta}_{jl}$. Because $m^{\xi}_{kl} < b$, we will have $m^{\xi}_{kl} < m^{\xi \Cup \eta}_{ki} + a + m^{\xi \Cup \eta}_{jl}$. Since $m^{\xi \Cup \eta}_{ki} \leq m^{\xi}_{ki}$ and $m^{\xi \Cup \eta}_{jl} \leq m^{\xi}_{jl}$, we will have $m^{\xi}_{kl} < m^{\xi}_{ki} + a + m^{\xi}_{jl}$.
   But $m^{\xi}_{ki} + m^{\xi}_{il} \leq m^{\xi}_{kl}$, so $m^{\xi}_{ki} + m^{\xi}_{il} < m^{\xi}_{ki} + a + m^{\xi}_{jl}$. Therefore $m^{\xi}_{il} < a + m^{\xi}_{jl}$. But $m^{\xi}_{ij} + m^{\xi}_{jl} \leq m^{\xi}_{il}$, and $m^{\xi}_{ij} = a$. So $a + m^{\xi}_{jl} < a + m^{\xi}_{jl}$: a contradiction.
   Because $m^{\xi \Cup \eta}_{ki} + a + m^{\xi \Cup \eta}_{jl} < b$, we will have $\boldsymbol{m}^{\xi \Cup \eta}_{kl} < b$.
   So $\boldsymbol{I}^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid \boldsymbol{m}^{\xi \Cup \eta}_{kl} \leq w < b\}$ will be non-empty.

If (i) or (ii).1, because ξ is consistent, we have $m^{\xi}_{ij} \leq m^{\xi}_{kl}$. Since $t_{ij} = a - \delta$, any value that is picked for t_{kl} from $\boldsymbol{I}^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl}$ will satisfy $t_{ij} \leq t_{kl}$.
If (ii).2, since $\boldsymbol{m}^{\xi \Cup \eta}_{kl} = m^{\xi \Cup \eta}_{ki} + t_{ij} + m^{\xi \Cup \eta}_{jl}$, we have $t_{ij} \leq \boldsymbol{m}^{\xi \Cup \eta}_{kl}$. Then any value t_{kl} in $\boldsymbol{I}^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid \boldsymbol{m}^{\xi \Cup \eta}_{kl} \leq w < b\}$ will satisfy $t_{ij} \leq t_{kl}$.
Obviously, $t_{ij} \in \boldsymbol{I}^{\xi \Cup \eta}_{ij}$ and $t_{kl} \in \boldsymbol{I}^{\xi \Cup \eta}_{il}$. Moreover, $0 \leq k < i < j < l < n$, so, by Observation 3, it is possible to form a sequence $t_k \leq t_i \leq t_j \leq t_l$ that is compatible with $\boldsymbol{D}^{\xi \Cup \eta}_{s}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \Cup \eta]\!]$. Clearly, $t^{\mathcal{B}^z}_{ij} \notin I^{\xi}_{ij}$ and $t^{\mathcal{B}^z}_{kl} \notin I^{\eta}_{kl}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl. □

The consequence of Theorem 3 is that if ξ and η have constraints that form a z_pair, then $\xi \cup \eta$ does not exist.

7 Conclusions

In our earlier work [13] we had found a sufficient condition for the existence of the union of two optimized timed scenarios $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$: if $\mathcal{C}_1$ and $\mathcal{C}_2$ do not contain a pair of constraints $\alpha = \tau_{i,j} \sim a$ and $\beta = \tau_{k,l} \sim b$, respectively, such that $i \neq k$ or $j \neq l$, $\alpha \notin \mathcal{C}_2$ and $\beta \notin \mathcal{C}_1$, then there will be no "zigzagging" behaviours between ξ and η, and therefore $\xi \cup \eta$ exists. Zigzagging behaviours belong neither to $[\![\xi]\!]$, nor to $[\![\eta]\!]$, even though they belong to $[\![\xi \Cup \eta]\!]$, i.e., the semantics of the combination (or the "quasi-union") of ξ and η.

In the current paper we investigate, in more depth, the conditions under which $[\![\xi \cup \eta]\!]$, where ξ and η are consistent, can be represented by a single scenario, namely the union $\xi \cup \eta$. Our investigation reveals that in the presence of zigzagging behaviours the constraints of ξ and η must satisfy certain additional criteria. Based on this observation we formulate a sufficient and necessary condition for the existence of the union (Theorem 2 and Theorem 3).

The union operation is directly relevant to the problem of synthesizing timed automata with minimal numbers of clocks from a set of scenarios [14]. It is also relevant to various model-checking techniques based on timed automata, if stable distance tables are used to capture zones corresponding to locations in the automata.

A detailed comparison of timed scenarios with other related work, in particular with Difference Bounds Matrices (DBMs) [24], can be found in our earlier work [10, 12]. A union operation has been defined for DBMs, which are used for representing zones in timed automata. However, the union of two zones (represented by DBMs) is—in general—a non-convex set, and therefore cannot be represented by a DBM. So the union of two zones has to be approximated, e.g., by convex hulls [18], or some other safe abstraction [17]. Another method for checking whether the union of two DBMs is itself a DBM has been developed using convex hulls together with Clock Difference Diagrams (CDDs) [25].

Our necessary and sufficient condition provides a syntactic criterion for the existence of the union of two timed scenarios. So the exact union can be computed when it exists. Otherwise, the combination (i.e., the "quasi-union") provides an approximation for the exact union.

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A Appendix: The complete proof of Theorem 1

Theorem 1. *Let ξ be a scenario of size n and $\mathcal{D}_s^\xi$ be its stable distance table. Let $\mathcal{J} = \{b_0, b_1, \ldots, b_k\} \subseteq \{i \in \mathbb{N} \mid 0 \leq i < n\}$, $k \geq 0$, $b_0 < b_1 < \cdots < b_k$ and let $S = t_{b_0} t_{b_1} \ldots t_{b_k}$ be compatible with $\mathcal{D}_s^\xi$. Then, for any $r \in \{i \in \mathbb{N} \mid 0 \leq i < n\} \setminus \mathcal{J}$, there exists a real number t_r such that*

1. *if $b_k < r < n$, then $t_{b_0} \ldots t_{b_k} \ldots t_r$ is compatible with $\mathcal{D}_s^\xi$;*
2. *if $b_0 < r < b_k$, then $t_{b_0} \ldots t_r \ldots t_{b_k}$ is compatible with $\mathcal{D}_s^\xi$;*
3. *if $0 \leq r < b_0$, then $t_r \ldots t_{b_0} \ldots t_{b_k}$ is compatible with $\mathcal{D}_s^\xi$.*

Proof. (Fig. 5 illustrates the three cases of the theorem.)
Case 1: If $k = 0$, then let $i = b_0$ and $b_0 < r$. We must show that there exists a real number t_r such that $t_i \leq t_r$ and $m_{ir} \leq t_{ir} \leq M_{ir}$. But because $m_{ir} \leq M_{ir}$, it is always possible to find a t_{ir} that satisfies the inequations.

Let $k > 0$, $b_k < r$ and let $i = b_p$ and $j = b_q$, where $0 \leq p < q \leq k$. We must show that there exists a real number t_r such that $t_i \leq t_j \leq t_r$ and

$$m_{jr} \leq t_{jr} \leq M_{jr} \quad (6) \qquad m_{ir} \leq t_{ir} \leq M_{ir} \quad (7)$$

Inequation (6) is equivalent to $m_{jr} \leq t_{ir} - t_{ij} \leq M_{jr}$, which is equivalent to

$$t_{ij} + m_{jr} \leq t_{ir} \leq t_{ij} + M_{jr} \tag{8}$$

Clearly, $t_{ij} + m_{jr} \leq t_{ij} + M_{jr}$, because $m_{jr} \leq M_{jr}$. Moreover, $0 \leq t_{ij} + m_{jr}$, because neither of the terms is negative. So it is possible to find a t_{ir} that satisfies (8), therefore (6) can be satisfied.

Inequation (7) is equivalent to $m_{ir} \leq t_{ij} + t_{jr} \leq M_{ir}$ which is equivalent to

$$m_{ir} - t_{ij} \leq t_{jr} \leq M_{ir} - t_{ij} \tag{9}$$

Obviously, $m_{ir} - t_{ij} \leq M_{ir} - t_{ij}$, because $m_{ir} \leq M_{ir}$.

It is easy to show that $0 \leq M_{ir} - t_{ij}$:

$\mathcal{D}_s^\xi$ is stable, so $M_{ij} + m_{jr} \leq M_{ir}$, therefore $m_{jr} \leq M_{ir} - M_{ij}$. Since $t_{ij} \leq M_{ij}$, $M_{ir} - M_{ij} \leq M_{ir} - t_{ij}$. But $0 \leq m_{jr}$, therefore $0 \leq M_{ir} - t_{ij}$.

So it is possible to find a t_{jr} that satisfies (9), therefore (7) can be satisfied.

Next, we must show that the same value of t_r can simultaneously satisfy both (6) and (7), or—equivalently—both (6) and (9). We do so by showing that $m_{jr} \leq M_{ir} - t_{ij}$ and $m_{ir} - t_{ij} \leq M_{jr}$.

$\mathcal{D}_s^\xi$ is stable, so $M_{ij} + m_{jr} \leq M_{ir}$, therefore $m_{jr} \leq M_{ir} - M_{ij} \leq M_{ir} - t_{ij}$ (because $t_{ij} \leq M_{ij}$).

Similarly, $m_{ir} \leq m_{ij} + M_{jr}$, therefore $m_{ir} - m_{ij} \leq M_{jr}$. But $m_{ir} - t_{ij} \leq m_{ir} - m_{ij}$ (because $m_{ij} \leq t_{ij}$), so $m_{ir} - t_{ij} \leq M_{jr}$.

Intuitively, this means that none of the lower bounds on t_r imposed by (6) and (7) exceeds any of the upper bounds imposed by these inequations, for any choice of i and j that satisfies the assumptions.

Case 2: Let $k > 0$, $i = b_p$ and $j = b_q$, where $0 \leq p < q \leq k$, and let $i < r < j$. We must show that there is a real number t_r such that $t_i \leq t_r \leq t_j$ and

$$m_{ir} \leq t_{ir} \leq M_{ir} \quad (10) \qquad m_{rj} \leq t_{rj} \leq M_{rj} \quad (11)$$

Inequation (10) is equivalent to $m_{ir} \leq t_{ij} - t_{rj} \leq M_{ir}$ which is equivalent to

$$t_{ij} - M_{ir} \leq t_{rj} \leq t_{ij} - m_{ir} \tag{12}$$

Obviously $t_{ij} - M_{ir} \leq t_{ij} - m_{ir}$, because $m_{ir} \leq M_{ir}$.

It is easy to show that $0 \leq t_{ij} - m_{ir}$:

$\mathcal{D}_s^\xi$ is stable, so $m_{ir} + m_{rj} \leq m_{ij}$, therefore $m_{rj} \leq m_{ij} - m_{ir} \leq t_{ij} - m_{ir}$ (because $m_{ij} \leq t_{ij}$). But $m_{rj} \geq 0$, therefore $t_{ij} - m_{ir} \geq 0$.

So it is possible to find a t_{rj} that satisfies (12), therefore (10) can be satisfied.

Inequation (11) is equivalent to $m_{rj} \leq t_{ij} - t_{ir} \leq M_{rj}$ which is equivalent to

$$t_{ij} - M_{rj} \leq t_{ir} \leq t_{ij} - m_{rj} \tag{13}$$

It is obvious that $t_{ij} - M_{rj} \leq t_{ij} - m_{rj}$ (because $m_{rj} \leq M_{rj}$).

It is easy to show that $0 \leq t_{ij} - m_{rj}$:

$\mathcal{D}_s^\xi$ is stable, so $m_{ir} + m_{rj} \leq m_{ij}$. Therefore, $m_{ir} \leq m_{ij} - m_{rj} \leq t_{ij} - m_{rj}$ (because $m_{ij} \leq t_{ij}$). But $0 \leq m_{ir}$, therefore $0 \leq t_{ij} - m_{rj}$.

So it is possible to find a t_{ir} that satisfies (13), therefore (11) can be satisfied.

Next, we must show that a single value of t_r can simultaneously satisfy both (10) and (11), or—equivalently—both (10) and (13). We do so by showing that $m_{ir} \leq t_{ij} - m_{rj}$ and $t_{ij} - M_{rj} \leq M_{ir}$.

$\mathcal{D}_s^\xi$ is stable, so $m_{ir} + m_{rj} \leq m_{ij}$, therefore $m_{ir} \leq m_{ij} - m_{rj} \leq t_{ij} - m_{rj}$ (because $m_{ij} \leq t_{ij}$).

Similarly, $M_{ij} \leq M_{ir} + M_{rj}$, therefore $M_{ij} - M_{rj} \leq M_{ir}$. Since $t_{ij} - M_{rj} \leq M_{ij} - M_{rj}$, we conclude that $t_{ij} - M_{rj} \leq M_{ir}$.

<u>Case 3:</u> Let $k > 0$ [6], $i = b_p$, $j = b_q$ for $0 \leq p < q \leq k$, and let $r < b_0$. We must show that there is a real number t_r such that $t_r \leq t_i \leq t_j$ and

$$m_{ri} \leq t_{ri} \leq M_{ri} \quad (14) \qquad m_{rj} \leq t_{rj} \leq M_{rj} \quad (15)$$

Inequation (14) is equivalent to $m_{ri} \leq t_{rj} - t_{ij} \leq M_{ri}$ which is equivalent to

$$m_{ri} + t_{ij} \leq t_{rj} \leq M_{ri} + t_{ij} \tag{16}$$

Clearly, $m_{ri} + t_{ij} \leq M_{ri} + t_{ij}$ (because $m_{ri} \leq M_{ri}$). Moreover, $0 \leq M_{ri} + t_{ij}$, because both m_{ri} and t_{ij} are not negative.

So it is possible to find a t_{rj} that satisfies (16), and therefore (14) can be satisfied.

Inequation (15) is equivalent to $m_{rj} \leq t_{ri} + t_{ij} \leq M_{rj}$ which is equivalent to

$$m_{rj} - t_{ij} \leq t_{ri} \leq M_{rj} - t_{ij} \tag{17}$$

Obviously, $m_{rj} - t_{ij} \leq M_{rj} - t_{ij}$ (because $m_{rj} \leq M_{rj}$).

It is easy to show that $0 \leq M_{rj} - t_{ij}$:

$\mathcal{D}_s^\xi$ is stable, so $m_{ri} + M_{ij} \leq M_{rj}$, therefore $m_{ri} \leq M_{rj} - M_{ij} \leq M_{rj} - t_{ij}$ (because $t_{ij} \leq M_{ij}$). But $0 \leq m_{ri}$, therefore $0 \leq M_{rj} - t_{ij}$.

So it is possible to find a t_{ri} that satisfies (17), therefore (15) can be satisfied.

Next, we must show that a single value of t_r can simultaneously satisfy both (14) and (15), or—equivalently—both (14) and (17). We do so by showing that $m_{ri} \leq M_{rj} - t_{ij}$ and $m_{rj} - t_{ij} \leq M_{ri}$.

$\mathcal{D}_s^\xi$ is stable, so $m_{ri} + M_{ij} \leq M_{rj}$, therefore $m_{ri} \leq M_{rj} - M_{ij} \leq M_{rj} - t_{ij}$ (because $t_{ij} \leq M_{ij}$).

[6] The proof for $k = 0$ is similar to that in Case 1, and is not presented.

Similarly, $m_{rj} \leq M_{ri} + m_{ij}$, therefore $m_{rj} - m_{ij} \leq M_{ri}$. Since $m_{rj} - t_{ij} \leq m_{rj} - m_{ij}$, we conclude that $m_{rj} - t_{ij} \leq M_{ri}$. □

B Appendix: The proof of Observation 3

Observation 3 *Let $\mathcal{D}$ be a stable distance table for a scenario of length n, $0 \leq i < j < n$, $0 \leq k < l < n$ and $i \neq k \vee j \neq l$. If I_{ij} and I_{kl} are collapsed intervals, then it is possible to construct a sequence $S = t_{b_0}t_{b_1}t_{b_2}t_{b_3}$ such that $\{b_0, b_1, b_2, b_3\} \subseteq \{i, j, k, l\}$ and S is compatible with $\mathcal{D}$.*

Proof. We show how to construct a compatible sequence for each of three[7] cases:

1. $0 \leq k \leq i < j \leq l < n$
2. $0 \leq i < k < j < l < n$
3. $0 \leq i < j \leq k < l < n$

- Case 1: $0 \leq k \leq i < j \leq l < n$ (see the diagrams in Fig. 6)
  Let $t_k \geq m_{0k}$, $t_i = t_k + m_{ki}$, $t_j = t_i + m_{ij}$ and $t_l = t_k + m_{kl}$.
  We must show that (a) $t_{ki} \in I_{ki}$, (b) $t_{kj} \in I_{kj}$, (c) $t_{il} \in I_{il}$ and (d) $t_{jl} \in I_{jl}$.
  (a) By definition, $t_{ki} = t_i - t_k = t_k + m_{ki} - t_k = m_{ki} \in I_{ki}$.
  (b) By definition, $t_{kj} = t_{ki} + t_{ij}$. Because I_{ij} is a collapsed interval, $t_{ij} = M_{ij}$. Therefore, $t_{kj} = m_{ki} + M_{ij}$. But $m_{kj} \leq m_{ki} + M_{ij} \leq M_{kj}$, so $t_{kj} \in I_{kj}$.
  (c) By definition, $t_{kl} = t_{ki} + t_{il}$. So $t_{il} = t_{kl} - t_{ki}$. Because I_{kl} is a collapsed interval, $t_{kl} = m_{kl}$. Therefore, $t_{il} = m_{kl} - m_{ki}$.
  We know $m_{ki} + m_{il} \leq m_{kl}$, so $m_{il} \leq m_{kl} - m_{ki}$. That is, $m_{il} \leq t_{il}$.
  We know $m_{kl} \leq m_{ki} + M_{il}$, so $m_{kl} - m_{ki} \leq M_{il}$. Therefore, $t_{il} \leq M_{il}$.
  (d) By definition, $t_{il} = t_{ij} + t_{jl}$, so $t_{jl} = t_{il} - t_{ij} = m_{kl} - m_{ki} - t_{ij}$.
  I_{ij} is a collapsed interval: $t_{ij} = m_{ij}$. So $t_{jl} = m_{kl} - (m_{ki} + m_{ij})$.
  We know $m_{ki} + m_{ij} \leq m_{kj}$, so, $t_{jl} \geq m_{kl} - m_{kj}$. But $m_{kj} + m_{jl} \leq m_{kl}$, so $m_{kl} - m_{kj} \geq m_{jl}$. Therefore, $t_{jl} \geq m_{jl}$.
  I_{ij} is a collapsed interval: $t_{ij} = M_{ij}$. So $t_{jl} = m_{kl} - (m_{ki} + M_{ij})$.
  We know $m_{kj} \leq m_{ki} + M_{ij}$, so, $t_{jl} \leq m_{kl} - m_{kj}$. But $m_{kl} \leq m_{kj} + M_{jl}$, so $m_{kl} - m_{kj} \leq M_{jl}$. Therefore, $t_{jl} \leq M_{jl}$.
- Case 2: $0 \leq i < k < j < l < n$ (see the diagram on the left of Fig. 7)
  Let $m_{0i} \leq t_i \leq M_{0i}$, $t_k = t_i + m_{ik}$, $t_j = t_i + m_{ij}$ and $t_l = t_k + m_{kl}$.
  We must show that (a) $t_{ik} \in I_{ik}$, (b) $t_{il} \in I_{il}$, (c) $t_{kj} \in I_{kj}$ and (d) $t_{jl} \in I_{jl}$.
  (a) By definition, $t_{ik} = t_k - t_i = t_i + m_{ik} - t_i = m_{ik} \in I_{ik}$.
  (b) By definition, $t_{il} = t_{ik} + t_{kl}$. Because I_{kl} is a collapsed interval, $t_{kl} = M_{kl}$. Therefore, $t_{il} = m_{ik} + M_{kl}$.
  We know $m_{il} \leq m_{ik} + M_{kl} \leq M_{il}$, so $t_{il} \in I_{il}$.
  (c) By definition, $t_{kj} = t_j - t_k = m_{ij} - m_{ik}$.
  We know $m_{ik} + m_{kj} \leq m_{ij}$, so $m_{kj} \leq m_{ij} - m_{ik}$. That is, $m_{kj} \leq t_{kj}$.
  Also, $m_{ij} \leq m_{ik} + M_{kj}$, so $m_{ij} - m_{ik} \leq M_{kj}$. Therefore, $t_{kj} \leq M_{kj}$.

[7] There are three other cases, which can be obtained from the above by interchanging i and j with k and l, but we do not need to consider them in the proof.

(d) By definition, $t_{jl} = t_l - t_j = t_k + m_{kl} - t_i - m_{ij} = m_{ik} + m_{kl} - m_{ij}$.
I_{kl} is a collapsed interval: $m_{kl} = M_{kl}$. So $t_{jl} = m_{ik} + M_{kl} - m_{ij}$.
We know $m_{il} \leq m_{ik} + M_{kl}$, so, $t_{jl} \geq m_{il} - m_{ij}$.
We know $m_{ij} + m_{jl} \leq m_{il}$, so $m_{il} - m_{ij} \geq m_{jl}$. Therefore, $t_{jl} \geq m_{jl}$.
I_{ij} is a collapsed interval: $m_{ij} = M_{ij}$. So, $t_{jl} = m_{ik} + m_{kl} - M_{ij}$.
We know $m_{ik} + m_{kl} \leq m_{il}$, so, $t_{jl} \leq m_{il} - M_{ij}$.
We know $m_{il} \leq M_{ij} + M_{jl}$, so, $m_{il} - M_{ij} \leq M_{jl}$. Therefore, $t_{jl} \leq M_{jl}$.

– Case 3: $0 \leq i < j \leq k < l < n$ (see the diagrams on the right of Fig. 7)
Let $m_{0i} \leq t_i \leq M_{0i}$. Let $t_j = t_i + m_{ij}$, $t_k = t_j + m_{jk}$ and $t_l = t_k + m_{kl}$.
We must show that (a) $t_{jk} \in I_{jk}$, (b) $t_{ik} \in I_{ik}$, (c) $t_{jl} \in I_{jl}$ and (d) $t_{il} \in I_{il}$.
  (a) By definition, $t_{jk} = t_k - t_j = t_j + m_{jk} - t_j = m_{jk} \in I_{jk}$.
  (b) By definition, $t_{ik} = t_{ij} + t_{jk}$. Because I_{ij} is a collapsed interval, $t_{ij} = M_{ij}$.
  Therefore, $t_{ik} = M_{ij} + m_{jk}$.
  We know $m_{ik} \leq M_{ij} + m_{jk} \leq M_{ik}$, so $t_{ik} \in I_{ik}$.
  (c) By definition, $t_{jl} = t_{jk} + t_{kl}$. Because I_{kl} is a collapsed interval, $t_{kl} = M_{kl}$.
  Therefore, $t_{jl} = m_{jk} + M_{kl}$.
  We know $m_{jl} \leq m_{jk} + M_{kl} \leq M_{jl}$, so $t_{jl} \in I_{jl}$.
  (d) By definition, $t_{il} = t_{ij} + t_{jk} + t_{kl} = M_{ij} + m_{jk} + t_{kl}$. Because I_{kl} is a collapsed interval, $t_{kl} = M_{kl}$. So $t_{il} = M_{ij} + m_{jk} + M_{kl}$.
  We know $m_{ik} \leq M_{ij} + m_{jk}$, so, $t_{il} \geq m_{ik} + M_{kl}$. But $m_{il} \leq m_{ik} + M_{kl}$, so $t_{il} \geq m_{il}$.
  Because I_{kl} is a collapsed interval, $t_{kl} = m_{kl}$. So $t_{il} = M_{ij} + m_{jk} + m_{kl}$.
  We know $M_{ij} + m_{jk} \leq M_{ik}$, so, $t_{il} \leq M_{ik} + m_{kl}$. But $M_{ik} + m_{kl} \leq M_{il}$, so $t_{il} \leq M_{il}$.

□

C Appendix: The complete proof of Theorem 2

Theorem 2. *Let ξ and η be two scenarios of length n with the same sequences of events, such that $\xi \not\subseteq \eta$ and $\eta \not\subseteq \xi$. If $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$ zigzags through ij and kl, then there exist $\alpha = \tau_{i,j} \sim a \in \mathcal{C}(\mathcal{D}_s^\xi)$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}(\mathcal{D}_s^\eta)$ ($a, b \in \mathbb{Q}$) such that α and β form a z_pair.*

Proof. **Case (1)**: $0 \leq k \leq i < j \leq l < n$.
Let $\mathcal{B}^z$ be a behaviour in $\mathcal{Z}(\xi, \eta)$ that zigzags through ij and kl. There are two cases to consider:

(1) $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^\xi$, $t_{ij}^{\mathcal{B}^z} \in I_{ij}^\eta$, $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^\eta$ and $t_{kl}^{\mathcal{B}^z} \in I_{kl}^\xi$.
  Since $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^\xi$, one of the following two cases must hold:
  (a) $t_{ij}^{\mathcal{B}^z} < m_{ij}^\xi$. Since $t_{ij}^{\mathcal{B}^z} \in I_{ij}^\eta$, we must have $m_{ij}^\eta \leq t_{ij}^{\mathcal{B}^z}$. It follows that $m_{ij}^\eta < m_{ij}^\xi$. Therefore $m_{ij}^\xi > 0$: there exists a non-default constraint $\alpha = \tau_{i,j} \geq a$ in $\mathcal{C}(\mathcal{D}_s^\xi)$, where $a = m_{ij}^\xi$.
  Since $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^\eta$, one of the following two cases must hold:

– $t^{\mathcal{B}^z}_{kl} < m^{\eta}_{kl}$. Since $t^{\mathcal{B}^z}_{kl} \in I^{\xi}_{kl}$, we must have $m^{\xi}_{kl} \leq t^{\mathcal{B}^z}_{kl}$. It follows that $m^{\xi}_{kl} < m^{\eta}_{kl}$. Therefore $m^{\eta}_{kl} > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}^{\eta}_s)$, where $b = m^{\eta}_{kl}$.
  So α and β form a z-pair (case (1)(a) of Definition 4).
– $M^{\eta}_{kl} < t^{\mathcal{B}^z}_{kl}$. Since $t^{\mathcal{B}^z}_{kl} \in I^{\xi}_{kl}$, we must have $t^{\mathcal{B}^z}_{kl} \leq M^{\xi}_{kl}$. It follows that $M^{\eta}_{kl} < M^{\xi}_{kl}$. Therefore $M^{\eta}_{kl} < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}^{\eta}_s)$, where $b = M^{\eta}_{kl}$.
  The constraints α and β fit case (1)(b) of Definition 4. We must show $M^{\xi \Cup \eta}_{ki} + a + M^{\xi \Cup \eta}_{jl} > b$.
  Assume $M^{\xi \Cup \eta}_{ki} + a + M^{\xi \Cup \eta}_{jl} \leq b$.
  By definition, $t^{\mathcal{B}^z}_{kl} = t^{\mathcal{B}^z}_{ki} + t^{\mathcal{B}^z}_{ij} + t^{\mathcal{B}^z}_{jl}$. Because $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$, we have $t^{\mathcal{B}^z}_{ki} \leq M^{\xi \Cup \eta}_{ki}$ and $t^{\mathcal{B}^z}_{jl} \leq M^{\xi \Cup \eta}_{jl}$. Therefore, $t^{\mathcal{B}^z}_{kl} \leq M^{\xi \Cup \eta}_{ki} + t^{\mathcal{B}^z}_{ij} + M^{\xi \Cup \eta}_{jl}$. Since $t^{\mathcal{B}^z}_{ij} < m^{\xi}_{ij}$ and $m^{\xi}_{ij} = a$, we have $t^{\mathcal{B}^z}_{kl} < M^{\xi \Cup \eta}_{ki} + a + M^{\xi \Cup \eta}_{jl}$. By the assumption, $M^{\xi \Cup \eta}_{ki} + a + M^{\xi \Cup \eta}_{jl} \leq b$. So it follows that $t^{\mathcal{B}^z}_{kl} < b$. But $M^{\eta}_{kl} < t^{\mathcal{B}^z}_{kl}$, so $M^{\eta}_{kl} < b$: a contradiction.
  So α and β form a z-pair (case (1)(b) of Definition 4).

(b) $M^{\xi}_{ij} < t^{\mathcal{B}^z}_{ij}$. Since $t^{\mathcal{B}^z}_{ij} \in I^{\eta}_{ij}$, we must have $t^{\mathcal{B}^z}_{ij} \leq M^{\eta}_{ij}$. It follows that $M^{\xi}_{ij} < M^{\eta}_{ij}$. Therefore $M^{\xi}_{ij} < \infty$: there exists a non-default constraint $\alpha = \tau_{i,j} \leq a$ in $\mathcal{C}(\mathcal{D}^{\xi}_s)$, where $a = M^{\xi}_{ij}$.
Since $t^{\mathcal{B}^z}_{kl} \notin I^{\eta}_{kl}$, one of the following two cases must hold:
– $t^{\mathcal{B}^z}_{kl} < m^{\eta}_{kl}$. Since $t^{\mathcal{B}^z}_{kl} \in I^{\xi}_{kl}$, we must have $m^{\xi}_{kl} \leq t^{\mathcal{B}^z}_{kl}$. It follows that $m^{\xi}_{kl} < m^{\eta}_{kl}$. Therefore $m^{\eta}_{kl} > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}^{\eta}_s)$, where $b = m^{\eta}_{kl}$.
  The constraints α and β fit case (1)(c) of Definition 4. We must show $m^{\xi \Cup \eta}_{ki} + a + m^{\xi \Cup \eta}_{jl} < b$.
  Assume $m^{\xi \Cup \eta}_{ki} + a + m^{\xi \Cup \eta}_{jl} \geq b$.
  By definition, $t^{\mathcal{B}^z}_{kl} = t^{\mathcal{B}^z}_{ki} + t^{\mathcal{B}^z}_{ij} + t^{\mathcal{B}^z}_{jl}$. Because $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$, we have $t^{\mathcal{B}^z}_{ki} \geq m^{\xi \Cup \eta}_{ki}$ and $t^{\mathcal{B}^z}_{jl} \geq m^{\xi \Cup \eta}_{jl}$. Therefore, $t^{\mathcal{B}^z}_{kl} \geq m^{\xi \Cup \eta}_{ki} + t^{\mathcal{B}^z}_{ij} + m^{\xi \Cup \eta}_{jl}$. Since $t^{\mathcal{B}^z}_{ij} > M^{\xi}_{ij}$ and $M^{\xi}_{ij} = a$, we have $t^{\mathcal{B}^z}_{kl} > m^{\xi \Cup \eta}_{ki} + a + m^{\xi \Cup \eta}_{jl}$. By the assumption, $m^{\xi \Cup \eta}_{ki} + a + m^{\xi \Cup \eta}_{jl} \geq b$. So it follows that $t^{\mathcal{B}^z}_{kl} > b$. But $m^{\eta}_{kl} > t^{\mathcal{B}^z}_{kl}$, so $m^{\eta}_{kl} > b$: a contradiction.
  So α and β form a z-pair (case (1)(c) of Definition 4).
– $M^{\eta}_{kl} < t^{\mathcal{B}^z}_{kl}$. Since $t^{\mathcal{B}^z}_{kl} \in I^{\xi}_{kl}$, we must have $t^{\mathcal{B}^z}_{kl} \leq M^{\xi}_{kl}$. It follows that $M^{\eta}_{kl} < M^{\xi}_{kl}$. Therefore $M^{\eta}_{kl} < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}^{\eta}_s)$, where $b = M^{\eta}_{kl}$.
  So α and β form a z-pair (case (1)(d) of Definition 4).

(2) $t^{\mathcal{B}^z}_{ij} \in I^{\xi}_{ij}$, $t^{\mathcal{B}^z}_{ij} \notin I^{\eta}_{ij}$, $t^{\mathcal{B}^z}_{kl} \in I^{\eta}_{kl}$ and $t^{\mathcal{B}^z}_{kl} \notin I^{\xi}_{kl}$.
The proof can be obtained by exchanging ξ and η in case (1).

<u>**Case (2)**</u>: $0 \leq i < k < j < l < n$.
Let $\mathcal{B}^z$ be a behaviour in $\mathcal{Z}(\xi, \eta)$ that zigzags through ij and kl. There are two cases to consider:

(1) $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$, $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\eta}$, $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$ and $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$.
Since $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$, one of the following two cases must hold:

(a) $t_{ij}^{\mathcal{B}^z} < m_{ij}^{\xi}$. Since $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\eta}$, we must have $m_{ij}^{\eta} \leq t_{ij}^{\mathcal{B}^z}$. It follows that $m_{ij}^{\eta} < m_{ij}^{\xi}$. Therefore $m_{ij}^{\xi} > 0$: there exists a non-default constraint $\alpha = \tau_{i,j} \geq a$ in $\mathcal{C}(\mathcal{D}_s^{\xi})$, where $a = m_{ij}^{\xi}$.
Since $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$, one of the following two cases must hold:

– $t_{kl}^{\mathcal{B}^z} < m_{kl}^{\eta}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$, we must have $m_{kl}^{\xi} \leq t_{kl}^{\mathcal{B}^z}$. It follows that $m_{kl}^{\xi} < m_{kl}^{\eta}$. Therefore $m_{kl}^{\eta} > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}_s^{\eta})$, where $b = m_{kl}^{\eta}$.
The constraints α and β fit case (2)(a) of Definition 4. We must show $m_{il}^{\xi \Cup \eta} - a < b - m_{kj}^{\xi \Cup \eta}$.
Assume $m_{il}^{\xi \Cup \eta} - a \geq b - m_{kj}^{\xi \Cup \eta}$, or equivalently $m_{il}^{\xi \Cup \eta} - a + m_{kj}^{\xi \Cup \eta} \geq b$. By definition, $t_{il}^{\mathcal{B}^z} = t_{ij}^{\mathcal{B}^z} + t_{jl}^{\mathcal{B}^z} = t_{ij}^{\mathcal{B}^z} + t_{kl}^{\mathcal{B}^z} - t_{kj}^{\mathcal{B}^z}$. So $t_{kl}^{\mathcal{B}^z} = t_{il}^{\mathcal{B}^z} - t_{ij}^{\mathcal{B}^z} + t_{kj}^{\mathcal{B}^z}$. Because $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$, we have $t_{il}^{\mathcal{B}^z} \geq m_{il}^{\xi \Cup \eta}$ and $t_{kj}^{\mathcal{B}^z} \geq m_{kj}^{\xi \Cup \eta}$. Therefore $t_{kl}^{\mathcal{B}^z} \geq m_{il}^{\xi \Cup \eta} - t_{ij}^{\mathcal{B}^z} + m_{kj}^{\xi \Cup \eta}$. Since $t_{ij}^{\mathcal{B}^z} < m_{ij}^{\xi}$ and $m_{ij}^{\xi} = a$, we have $t_{kl}^{\mathcal{B}^z} > m_{il}^{\xi \Cup \eta} - a + m_{kj}^{\xi \Cup \eta}$. By the assumption, $m_{il}^{\xi \Cup \eta} - a + m_{kj}^{\xi \Cup \eta} \geq b$. So it follows that $t_{kl}^{\mathcal{B}^z} > b$. But $t_{kl}^{\mathcal{B}^z} < m_{kl}^{\eta}$, so $m_{kl}^{\eta} > b$: a contradiction. So α and β form a z-pair (case (2)(a) of Definition 4).

– $M_{kl}^{\eta} < t_{kl}^{\mathcal{B}^z}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$, we must have $t_{kl}^{\mathcal{B}^z} \leq M_{kl}^{\xi}$. It follows that $M_{kl}^{\eta} < M_{kl}^{\xi}$. Therefore $M_{kl}^{\eta} < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}_s^{\eta})$, where $b = M_{kl}^{\eta}$.
The constraints α and β fit case (2)(b) of Definition 4. We must show $a + M_{jl}^{\xi \Cup \eta} > m_{ik}^{\xi \Cup \eta} + b$.
Assume $a + M_{jl}^{\xi \Cup \eta} \leq m_{ik}^{\xi \Cup \eta} + b$, or equivalently, $a + M_{jl}^{\xi \Cup \eta} - m_{ik}^{\xi \Cup \eta} \leq b$. By definition, $t_{ij}^{\mathcal{B}^z} - t_{ik}^{\mathcal{B}^z} = t_{kl}^{\mathcal{B}^z} - t_{jl}^{\mathcal{B}^z}$, so $t_{kl}^{\mathcal{B}^z} = t_{ij}^{\mathcal{B}^z} + t_{jl}^{\mathcal{B}^z} - t_{ik}^{\mathcal{B}^z}$. Because $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$, we have $M_{ik}^{\xi \Cup \eta} \leq t_{ik}^{\mathcal{B}^z}$ and $t_{jl}^{\mathcal{B}^z} \leq M_{jl}^{\xi \Cup \eta}$. Therefore $t_{kl}^{\mathcal{B}^z} \leq t_{ij}^{\mathcal{B}^z} + M_{jl}^{\xi \Cup \eta} - m_{ik}^{\xi \Cup \eta}$. Since $t_{ij}^{\mathcal{B}^z} < m_{ij}^{\xi}$ and $m_{ij}^{\xi} = a$, we have $t_{kl}^{\mathcal{B}^z} < a + M_{jl}^{\xi \Cup \eta} - m_{ik}^{\xi \Cup \eta}$. By the assumption, $a + M_{jl}^{\xi \Cup \eta} - m_{ik}^{\xi \Cup \eta} \leq b$. So it follows that $t_{kl}^{\mathcal{B}^z} < b$. But $M_{kl}^{\eta} < t_{kl}^{\mathcal{B}^z}$, so $M_{kl}^{\eta} < b$: a contradiction. So α and β form a z-pair (case (2)(b) of Definition 4).

(b) $M_{ij}^{\xi} < t_{ij}^{\mathcal{B}^z}$. Since $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\eta}$, we must have $t_{ij}^{\mathcal{B}^z} \leq M_{ij}^{\eta}$. It follows that $M_{ij}^{\xi} < M_{ij}^{\eta}$. Therefore $M_{ij}^{\xi} < \infty$: there exists a non-default constraint $\alpha = \tau_{i,j} \leq a$ in $\mathcal{C}(\mathcal{D}_s^{\xi})$, where $a = M_{ij}^{\xi}$.
Since $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$, one of the following two cases must hold:

– $t_{kl}^{\mathcal{B}^z} < m_{kl}^{\eta}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$, we must have $m_{kl}^{\xi} \leq t_{kl}^{\mathcal{B}^z}$. It follows that $m_{kl}^{\xi} < m_{kl}^{\eta}$. Therefore $m_{kl}^{\eta} > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}_s^{\eta})$, where $b = m_{kl}^{\eta}$.
The constraints α and β fit case (2)(c) of Definition 4. We must show $a + m_{jl}^{\xi \Cup \eta} < M_{ik}^{\xi \Cup \eta} + b$.
Assume $a + m_{jl}^{\xi \Cup \eta} \geq M_{ik}^{\xi \Cup \eta} + b$, or equivalently, $a + m_{jl}^{\xi \Cup \eta} - M_{ik}^{\xi \Cup \eta} \geq b$.

By definition, $t^{\mathcal{B}^z}_{ij} - t^{\mathcal{B}^z}_{ik} = t^{\mathcal{B}^z}_{kl} - t^{\mathcal{B}^z}_{jl}$, so $t^{\mathcal{B}^z}_{kl} = t^{\mathcal{B}^z}_{ij} + t^{\mathcal{B}^z}_{jl} - t^{\mathcal{B}^z}_{ik}$. Because $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$, we have $M^{\xi\Cup\eta}_{ik} \geq t^{\mathcal{B}^z}_{ik}$ and $t^{\mathcal{B}^z}_{jl} \geq m^{\xi\Cup\eta}_{jl}$. Therefore, $t^{\mathcal{B}^z}_{kl} \geq t^{\mathcal{B}^z}_{ij} + m^{\xi\Cup\eta}_{jl} - M^{\xi\Cup\eta}_{ik}$. Since $t^{\mathcal{B}^z}_{ij} > M^{\xi}_{ij}$ and $M^{\xi}_{ij} = a$, we have $t^{\mathcal{B}^z}_{kl} > a + m^{\xi\Cup\eta}_{jl} - M^{\xi\Cup\eta}_{ik}$. By the assumption, $a + m^{\xi\Cup\eta}_{jl} - M^{\xi\Cup\eta}_{ik} \geq b$. So it follows that $t^{\mathcal{B}^z}_{kl} > b$. But $m^{\eta}_{kl} > t^{\mathcal{B}^z}_{kl}$, so $m^{\eta}_{kl} > b$: a contradiction.
So α and β form a z-pair (case (2)(c) of Definition 4).

– $M^{\eta}_{kl} < t^{\mathcal{B}^z}_{kl}$. Since $t^{\mathcal{B}^z}_{kl} \in I^{\xi}_{kl}$, we must have $t^{\mathcal{B}^z}_{kl} \leq M^{\xi}_{kl}$. It follows that $M^{\eta}_{kl} < M^{\xi}_{kl}$. Therefore $M^{\eta}_{kl} < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}^{\eta}_s)$, where $b = M^{\eta}_{kl}$.
Constraints α and β fit case (2)(d) of Definition 4. We must show $M^{\xi\Cup\eta}_{il} - a > b - M^{\xi\Cup\eta}_{kj}$.
Assume $M^{\xi\Cup\eta}_{il} - a \leq b - M^{\xi\Cup\eta}_{kj}$, or equivalently $M^{\xi\Cup\eta}_{il} - a + M^{\xi\Cup\eta}_{kj} \leq b$. By definition, $t^{\mathcal{B}^z}_{il} = t^{\mathcal{B}^z}_{ij} + t^{\mathcal{B}^z}_{jl} = t^{\mathcal{B}^z}_{ij} + t^{\mathcal{B}^z}_{kl} - t^{\mathcal{B}^z}_{kj}$. So $t^{\mathcal{B}^z}_{kl} = t^{\mathcal{B}^z}_{il} - t^{\mathcal{B}^z}_{ij} + t^{\mathcal{B}^z}_{kj}$. Because $\mathcal{B}^z \in [\![\xi \Cup \eta]\!]$, we have $t^{\mathcal{B}^z}_{il} \leq M^{\xi\Cup\eta}_{il}$ and $t^{\mathcal{B}^z}_{kj} \leq M^{\xi\Cup\eta}_{kj}$. Therefore $t^{\mathcal{B}^z}_{kl} \leq M^{\xi\Cup\eta}_{il} - t^{\mathcal{B}^z}_{ij} + M^{\xi\Cup\eta}_{kj}$. Since $M^{\xi}_{ij} < t^{\mathcal{B}^z}_{ij}$ and $M^{\xi}_{ij} = a$, we have $t^{\mathcal{B}^z}_{kl} < M^{\xi\Cup\eta}_{il} - a + M^{\xi\Cup\eta}_{kj}$. By the assumption, $M^{\xi\Cup\eta}_{il} - a + M^{\xi\Cup\eta}_{kj} \leq b$. So it follows that $t^{\mathcal{B}^z}_{kl} < b$. But $M^{\eta}_{kl} < t^{\mathcal{B}^z}_{kl}$, so $M^{\eta}_{kl} < b$: a contradiction.
So α and β form a z-pair (case (2)(d) of Definition 4).

(2) $t^{\mathcal{B}^z}_{ij} \in I^{\xi}_{ij}$, $t^{\mathcal{B}^z}_{ij} \notin I^{\eta}_{ij}$, $t^{\mathcal{B}^z}_{kl} \in I^{\eta}_{kl}$ and $t^{\mathcal{B}^z}_{kl} \notin I^{\xi}_{kl}$.
The proof can be obtained by exchanging ξ and η in case (2).

<u>**Case (3)**</u>: $0 \leq i < j \leq k < l < n$.
Let $\mathcal{B}^z$ be a behaviour in $\mathcal{Z}(\xi, \eta)$ that zigzags through ij and kl. There are two cases to consider:

(1) $t^{\mathcal{B}^z}_{ij} \notin I^{\xi}_{ij}$, $t^{\mathcal{B}^z}_{ij} \in I^{\eta}_{ij}$, $t^{\mathcal{B}^z}_{kl} \notin I^{\eta}_{kl}$ and $t^{\mathcal{B}^z}_{kl} \in I^{\xi}_{kl}$.
Since $t^{\mathcal{B}^z}_{ij} \notin I^{\xi}_{ij}$, one of the following two cases must hold:
(a) $t^{\mathcal{B}^z}_{ij} < m^{\xi}_{ij}$. Since $t^{\mathcal{B}^z}_{ij} \in I^{\eta}_{ij}$, we must have $m^{\eta}_{ij} \leq t^{\mathcal{B}^z}_{ij}$. It follows that $m^{\eta}_{ij} < m^{\xi}_{ij}$. Therefore $m^{\xi}_{ij} > 0$: there exists a non-default constraint $\alpha = \tau_{i,j} \geq a$ in $\mathcal{C}(\mathcal{D}^{\xi}_s)$, where $a = m^{\xi}_{ij}$.
Since $t^{\mathcal{B}^z}_{kl} \notin I^{\eta}_{kl}$, one of the following two cases must hold:
– $t^{\mathcal{B}^z}_{kl} < m^{\eta}_{kl}$. Since $t^{\mathcal{B}^z}_{kl} \in I^{\xi}_{kl}$, we must have $m^{\xi}_{kl} \leq t^{\mathcal{B}^z}_{kl}$. It follows that $m^{\xi}_{kl} < m^{\eta}_{kl}$. Therefore $m^{\eta}_{kl} > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}^{\eta}_s)$, where $b = m^{\eta}_{kl}$.
So α and β form a z-pair (case (3)(a) of Definition 4).
– $M^{\eta}_{kl} < t^{\mathcal{B}^z}_{kl}$. Since $t^{\mathcal{B}^z}_{kl} \in I^{\xi}_{kl}$, we must have $t^{\mathcal{B}^z}_{kl} \leq M^{\xi}_{kl}$. It follows that $M^{\eta}_{kl} < M^{\xi}_{kl}$. Therefore $M^{\eta}_{kl} < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}^{\eta}_s)$, where $b = M^{\eta}_{kl}$.
So α and β form a z-pair (case (3)(b) of Definition 4).

(b) $M^{\xi}_{ij} < t^{\mathcal{B}^z}_{ij}$. Since $t^{\mathcal{B}^z}_{ij} \in I^{\eta}_{ij}$, we must have $t^{\mathcal{B}^z}_{ij} \leq M^{\eta}_{ij}$. It follows that $M^{\xi}_{ij} < M^{\eta}_{ij}$. Therefore $M^{\xi}_{ij} < \infty$: there exists a non-default constraint $\alpha = \tau_{i,j} \leq a$ in $\mathcal{C}(\mathcal{D}^{\xi}_s)$, where $a = M^{\xi}_{ij}$.

Since $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$, one of the following two cases must hold:

- $t_{kl}^{\mathcal{B}^z} < m_{kl}^{\eta}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$, we must have $m_{kl}^{\xi} \leq t_{kl}^{\mathcal{B}^z}$. It follows that $m_{kl}^{\xi} < m_{kl}^{\eta}$. Therefore $m_{kl}^{\eta} > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}_s^{\eta})$, where $b = m_{kl}^{\eta}$.
  So α and β form a z-pair (case (3)(c) of Definition 4).
- $M_{kl}^{\eta} < t_{kl}^{\mathcal{B}^z}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\xi}$, we must have $t_{kl}^{\mathcal{B}^z} \leq M_{kl}^{\xi}$. It follows that $M_{kl}^{\eta} < M_{kl}^{\xi}$. Therefore $M_{kl}^{\eta} < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}_s^{\eta})$, where $b = M_{kl}^{\eta}$.
  So α and β form a z-pair (case (3)(d) of Definition 4).

(2) $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\xi}$, $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\eta}$, $t_{kl}^{\mathcal{B}^z} \in I_{kl}^{\eta}$ and $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\xi}$.
The proof can be obtained by exchanging ξ and η in case (3).

□

D Appendix: The proof of Observation 4

Observation 4 *Let $\mathcal{D}$ be a stable distance table for a scenario of length n and let $0 \leq i \leq j \leq k < n$ be integers.*

Let t_{ij} satisfy $m_{ij} \leq t_{ij} \leq M_{ij}$. Then, after replacing m_{ij} and M_{ij} with t_{ij} and stabilising the table, if m_{ik} changes, it will increase to at most $t_{ij} + m_{jk}$.

Proof. (We will use bold font to represent values in the modified table.)

After m_{ij} and M_{ij} have been replaced with t_{ij}, i.e., $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$, there are three possibilities for m_{ik} to potentially change. We explore the possibilities in the following order[8]:

1. The inequation $m_{ik'} \leq m_{ik} + M_{kk'}$, for some $k < k'$, does not hold. Then there are two cases:
   (a) The inequation $m_{ik'} \leq m_{ik} + M_{kk'}$ does not hold because $M_{kk'}$ has changed. If that is the case, then $\boldsymbol{M}_{kk'} < M_{kk'}$, and the inequation will be restored by setting $\boldsymbol{m}_{ik} := m_{ik'} - \boldsymbol{M}_{kk'}$.
   But for this to happen, after setting $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$, the inequation $m_{ik} + M_{kk'} \leq M_{ik'}$ has ceased to hold for the new values of some of these three variables. That is, $m_{ik} + M_{kk'} > M_{ik'}$. There are two reasons that the inequation could not hold:
   - m_{ik} has increased. But this is not the case: by the assumption, m_{ik} has not changed yet.
   - After setting $t_{ij} = M_{ij} = m_{ij}$, $M_{ik'}$ has changed. But in that case M_{ik} must have changed first: $\boldsymbol{M}_{ik} := t_{ij} + M_{jk}$. As a result of this change the inequation $M_{ik'} \leq \boldsymbol{M}_{ik} + M_{kk'}$ does not hold, that is, $\boldsymbol{M}_{ik} + M_{kk'} < M_{ik'}$. But because $m_{ik} \leq \boldsymbol{M}_{ik}$, it follows that $m_{ik} + M_{kk'} < M_{ik'}$. But because $M_{ik'} < m_{ik} + M_{kk'}$, it follows that $M_{ik'} < M_{ik'}$: a contradiction.

[8] We know the order in which the rules of stabilisation are applied does not affect the outcome (confluence) [11].

(b) The inequation $m_{ik'} \leq m_{ik} + M_{kk'}$ does not hold because $m_{ik'}$ has changed. If that is the case, then $m_{ik'} < \boldsymbol{m}_{ik'}$, and the inequation will be restored by setting $\boldsymbol{m}_{ik} := \boldsymbol{m}_{ik'} - M_{kk'}$.
But after setting $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$, it would be impossible for $m_{ik'}$ to change before m_{ik} is changed.

2. There exists some $i' < i$ such that $m_{i'k} \leq M_{i'i} + m_{ik}$ does not hold. Then there are two cases:
   (a) The inequation $m_{i'k} \leq M_{i'i} + m_{ik}$ does not hold because $m_{i'k}$ has changed. If that is the case, then $m_{i'k} < \boldsymbol{m}_{i'k}$, and the inequation will be restored by setting $\boldsymbol{m}_{ik} := \boldsymbol{m}_{i'k} - M_{i'i}$.
   But for this to happen, after setting $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$, one of the following three things must have happened:
   - The inequation $m_{i'i} + m_{ik} \leq m_{i'k}$ did not hold. But this is not possible because m_{ik} has not changed yet.
   - Inequation $m_{i''k} \leq M_{i''i'} + m_{i'k}$, for some $i'' < i'$, did not hold. But it would be impossible for $m_{i''k}$ to change before m_{ik} is changed.
   - Inequation $m_{i'k'} \leq m_{i'k} + M_{kk'}$, for some $k < k'$, did not hold. But it would be impossible for $m_{i'k'}$ to change before m_{ik} being changed.

   (b) The inequation $m_{i'k} \leq M_{i'i} + m_{ik}$ does not hold because $M_{i'i}$ has changed. If that is the case, then $\boldsymbol{M}_{i'i} < M_{i'i}$, and the inequation will be restored by setting $\boldsymbol{m}_{ik} := m_{i'k} - \boldsymbol{M}_{i'i}$.
   For this to happen, after setting $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$, the inequation $M_{i'i} + \boldsymbol{m}_{ij} \leq M_{i'j}$ has ceased to hold for the new values of some of these three variables. Then the inequation was restored by setting $\boldsymbol{M}_{i'i} := M_{i'j} - t_{ij}$.
   Then, $\boldsymbol{m}_{ik} := m_{i'k} - (M_{i'j} - t_{ij}) = m_{i'k} - M_{i'j} + t_{ij}$.
   We show that $\boldsymbol{m}_{ik} \leq t_{ij} + m_{jk}$.
   We have $m_{i'k} \leq M_{i'j} + m_{jk}$. Therefore, $m_{i'k} - M_{i'j} \leq m_{jk}$. It follows that $\boldsymbol{m}_{ik} \leq t_{ij} + m_{jk}$.
3. The inequation $\boldsymbol{m}_{ij} + m_{jk} \leq m_{ik}$ does not hold. In that case the inequation is restored by setting $m_{ik} := t_{ij} + m_{jk}$.

□

E Appendix: The complete proof of Theorem 3

Observation 6 *Let $\mathcal{D}$ be a stable distance table for a scenario of length n and let $0 \leq i \leq j \leq k < n$ be integers.*

Let t_{ij} satisfy $m_{ij} \leq t_{ij} \leq M_{ij}$. Then, after replacing m_{ij} and M_{ij} with t_{ij} and stabilising the table, if M_{ik} changes, it will increase to at most $t_{ij} + M_{jk}$.

Proof. The proof is very similar to the proof of Observation 4.

Observation 7 *Let $\mathcal{D}$ be a stable distance table for a scenario of length n and let $0 \leq i \leq j \leq k < n$ be integers. Let t_{ij} satisfy $m_{ij} \leq t_{ij} \leq M_{ij}$. Then, after replacing m_{ij} and M_{ij} with t_{ij}, if the inequation $M_{ik} \leq t_{ij} + M_{jk}$ holds, then during stabilisation M_{ik} will not change.*

Proof. The proof is very similar to the proof of Observation 5.

Theorem 3. *Let ξ and η be two scenarios of length n with the same sequences of events, such that $\xi \not\sqsubseteq \eta$, $\eta \not\sqsubseteq \xi$ and $\xi \Cup \eta$ is defined. If there are $\alpha = \tau_{i,j} \sim a \in \mathcal{C}(\mathcal{D}^{\xi}_{s})$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}(\mathcal{D}^{\eta}_{s})$ $(a, b \in \mathbb{Q})$, such that α and β form a z_pair, then there is a $\mathcal{B}^{z} \in \mathcal{Z}(\xi, \eta)$, such that $\mathcal{B}^{z}$ zigzags through ij and kl.*

Proof. Just as in Definition 4, there are three cases, determined by the relative positions of i, j, k and l. Each of these cases has four subcases, determined by the relations between the minima and maxima of the relevant intervals.

Case (1): $0 \le k \le i < j \le l < n$ $(i \ne k \vee j \ne l)$.

We show the proof for the case $0 \le k < i < j < l < n$. The proofs for $k = i$ or $j = l$ are special cases of the proof shown below.

We must show that there is a behaviour $\mathcal{B}^{z} \in [\![\xi \Cup \eta]\!]$ such that $\mathcal{B}^{z}$ zigzags through ij and kl. Because ij is entirely within kl, i.e., $k < i < j < l$, we must have $t^{\mathcal{B}^{z}}_{ij} \le t^{\mathcal{B}^{z}}_{kl}$.

(a) $\alpha = \tau_{i,j} \ge a$, $\beta = \tau_{k,l} \ge b$, $m^{\eta}_{ij} < a$, $m^{\xi}_{kl} < b$

Because of constraint α, $m^{\xi}_{ij} = a$ and because of constraint β, $m^{\eta}_{kl} = b$. $m^{\xi\Cup\eta}_{ij} = \min(m^{\xi}_{ij}, m^{\eta}_{ij}) = m^{\eta}_{ij}$ and $m^{\xi\Cup\eta}_{kl} = \min(m^{\xi}_{kl}, m^{\eta}_{kl}) = m^{\xi}_{kl}$.
Both $I^{\xi\Cup\eta}_{ij} \setminus I^{\xi}_{ij} = \{u \in \mathbb{Q}_{>0} \mid m^{\eta}_{ij} \le u < a\}$ and $I^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\ge 0} \mid m^{\xi}_{kl} \le w < b\}$ are non-empty. So it is possible to pick a value $t_{ij} = a - \delta$ $(\delta \in \mathbb{Q}_{>0})$ in the interval $I^{\xi\Cup\eta}_{ij} \setminus I^{\xi}_{ij}$.
Next we must show that after collapsing $I^{\xi\Cup\eta}_{ij}$ of $\mathcal{D}^{\xi\Cup\eta}_{s}$ to t_{ij} (i.e., setting $m^{\xi\Cup\eta}_{ij} = M^{\xi\Cup\eta}_{ij} = t_{ij}$ and stabilizing $\mathcal{D}^{\xi\Cup\eta}_{s}$), hence obtaining $\boldsymbol{D}^{\xi\Cup\eta}_{s}$, it is still possible to pick a value t_{kl} in the interval $\boldsymbol{I}^{\xi\Cup\eta}_{kl}$ of table $\boldsymbol{D}^{\xi\Cup\eta}_{s}$, such that t_{kl} belongs to I^{ξ}_{kl}, but does not belong to I^{η}_{kl}, and that $t_{ij} \le t_{kl}$.
(We use bold font to distinguish items that are updated during stabilization.)
In $\mathcal{D}^{\xi\Cup\eta}_{s}$ we have $m^{\xi\Cup\eta}_{ki} + m^{\xi\Cup\eta}_{ij} \le m^{\xi\Cup\eta}_{kj}$ and $m^{\xi\Cup\eta}_{kj} + m^{\xi\Cup\eta}_{jl} \le m^{\xi\Cup\eta}_{kl}$.
We set $\boldsymbol{m}^{\xi\Cup\eta}_{ij} = \boldsymbol{M}^{\xi\Cup\eta}_{ij} = t_{ij}$ in interval $\boldsymbol{I}^{\xi\Cup\eta}_{ij}$ of $\mathcal{D}^{\xi\Cup\eta}_{s}$ stabilise it to obtain $\boldsymbol{D}^{\xi\Cup\eta}_{s}$. After this one of the following two things will happen:

(i) The inequation $m^{\xi\Cup\eta}_{ki} + t_{ij} \le m^{\xi\Cup\eta}_{kj}$ holds, in which case, by Observation 5, $m^{\xi\Cup\eta}_{kj}$ will not change: $\boldsymbol{m}^{\xi\Cup\eta}_{kj} = m^{\xi\Cup\eta}_{kj}$. The inequation $m^{\xi\Cup\eta}_{kj} + m^{\xi\Cup\eta}_{jl} \le m^{\xi\Cup\eta}_{kl}$ will then continue to hold. Therefore $m^{\xi\Cup\eta}_{kl}$ will not change: $\boldsymbol{m}^{\xi\Cup\eta}_{kl} = m^{\xi\Cup\eta}_{kl} = m^{\xi}_{kl}$. Hence, $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\ge 0} \mid m^{\xi}_{kl} \le w < b\}$.

(ii) The inequation $m^{\xi\Cup\eta}_{ki} + t_{ij} \le m^{\xi\Cup\eta}_{kj}$ does not hold. In that case, to restore the inequation, $m^{\xi\Cup\eta}_{kj}$ will be set to $m^{\xi\Cup\eta}_{ki} + t_{ij}$. By Observation 4, $m^{\xi\Cup\eta}_{kj}$ cannot further increase, therefore $\boldsymbol{m}^{\xi\Cup\eta}_{kj} = m^{\xi\Cup\eta}_{ki} + a - \delta$. But since $\boldsymbol{m}^{\xi\Cup\eta}_{kj}$ has changed, the inequation $\boldsymbol{m}^{\xi\Cup\eta}_{kj} + m^{\xi\Cup\eta}_{jl} \le m^{\xi\Cup\eta}_{kl}$ might not hold. There are two cases to consider:

1. If the new value of $\boldsymbol{m}^{\xi\Cup\eta}_{kj}$ does not affect the satisfiability of the inequation $\boldsymbol{m}^{\xi\Cup\eta}_{kj} + m^{\xi\Cup\eta}_{jl} \le m^{\xi\Cup\eta}_{kl}$, then $m^{\xi\Cup\eta}_{kl}$ will remain the same, so just as in case (i), $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\ge 0} \mid m^{\xi}_{kl} \le w < b\}$.

2. If $\boldsymbol{m}^{\xi\Cup\eta}_{kj} + m^{\xi\Cup\eta}_{jl} \leq m^{\xi\Cup\eta}_{kl}$ is not satisfied, the inequation will be restored by setting $m^{\xi\Cup\eta}_{kl}$ to $\boldsymbol{m}^{\xi\Cup\eta}_{kj} + m^{\xi\Cup\eta}_{jl}$: $\boldsymbol{m}^{\xi\Cup\eta}_{kl} := \boldsymbol{m}^{\xi\Cup\eta}_{kj} + m^{\xi\Cup\eta}_{jl} = m^{\xi\Cup\eta}_{ki} + a - \delta + m^{\xi\Cup\eta}_{jl}$. Now the inequation $\boldsymbol{m}^{\xi\Cup\eta}_{kj} + m^{\xi\Cup\eta}_{jl} \leq \boldsymbol{m}^{\xi\Cup\eta}_{kl}$ holds, and by Observation 5, $\boldsymbol{m}^{\xi\Cup\eta}_{kl}$ cannot be further increased. Next, we show that $\boldsymbol{m}^{\xi\Cup\eta}_{kl} < b$. For that, we show that $m^{\xi\Cup\eta}_{ki} + a + m^{\xi\Cup\eta}_{jl} < b$.
   Assume $b \leq m^{\xi\Cup\eta}_{ki} + a + m^{\xi\Cup\eta}_{jl}$. Because $m^{\xi}_{kl} < b$, we will have $m^{\xi}_{kl} < m^{\xi\Cup\eta}_{ki} + a + m^{\xi\Cup\eta}_{jl}$. Since $m^{\xi\Cup\eta}_{ki} \leq m^{\xi}_{ki}$ and $m^{\xi\Cup\eta}_{jl} \leq m^{\xi}_{jl}$, we will have $m^{\xi}_{kl} < m^{\xi}_{ki} + a + m^{\xi}_{jl}$.
   But $m^{\xi}_{ki} + m^{\xi}_{il} \leq m^{\xi}_{kl}$, so $m^{\xi}_{ki} + m^{\xi}_{il} < m^{\xi}_{ki} + a + m^{\xi}_{jl}$. Therefore $m^{\xi}_{il} < a + m^{\xi}_{jl}$. But $m^{\xi}_{ij} + m^{\xi}_{jl} \leq m^{\xi}_{il}$, and $m^{\xi}_{ij} = a$. So $a + m^{\xi}_{jl} < a + m^{\xi}_{jl}$: a contradiction.
   Because $m^{\xi\Cup\eta}_{ki} + a + m^{\xi\Cup\eta}_{jl} < b$, we will have $\boldsymbol{m}^{\xi\Cup\eta}_{kl} < b$.
   So $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid \boldsymbol{m}^{\xi\Cup\eta}_{kl} \leq w < b\}$ will be non-empty.

If (i) or (ii).1, because ξ is consistent, we have $m^{\xi}_{ij} \leq m^{\xi}_{kl}$. Since $t_{ij} = a - \delta$, any value that is picked for t_{kl} from $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl}$ will satisfy $t_{ij} \leq t_{kl}$.
If (ii).2, since $\boldsymbol{m}^{\xi\Cup\eta}_{kl} = m^{\xi\Cup\eta}_{ki} + t_{ij} + m^{\xi\Cup\eta}_{jl}$, we have $t_{ij} \leq \boldsymbol{m}^{\xi\Cup\eta}_{kl}$. Then any value t_{kl} in $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid \boldsymbol{m}^{\xi\Cup\eta}_{kl} \leq w < b\}$ will satisfy $t_{ij} \leq t_{kl}$.
Obviously, $t_{ij} \in \boldsymbol{I}^{\xi\Cup\eta}_{ij}$ and $t_{kl} \in \boldsymbol{I}^{\xi\Cup\eta}_{il}$. Moreover, $0 \leq k < i < j < l < n$, so, by Observation 3, it is possible to form a sequence $t_k \leq t_i \leq t_j \leq t_l$ that is compatible with $\boldsymbol{D}^{\xi\Cup\eta}_{s}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \Cup \eta]\!]$. Clearly, $t^{\mathcal{B}^z}_{ij} \notin I^{\xi}_{ij}$ and $t^{\mathcal{B}^z}_{kl} \notin I^{\eta}_{kl}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl.

(b) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \leq b$, $m^{\eta}_{ij} < a$, $b < M^{\xi}_{kl}$ and $M^{\xi\Cup\eta}_{ki} + a + M^{\xi\Cup\eta}_{jl} > b$.

Because of constraint α, $m^{\xi}_{ij} = a$ and because of constraint β, $M^{\eta}_{kl} = b$. $m^{\xi\Cup\eta}_{ij} = \min(m^{\xi}_{ij}, m^{\eta}_{ij}) = m^{\eta}_{ij}$ and $M^{\xi\Cup\eta}_{kl} = \max(M^{\xi}_{kl}, M^{\eta}_{kl}) = M^{\xi}_{kl}$.
Initially, both $I^{\xi\Cup\eta}_{ij} \setminus I^{\xi}_{ij} = \{u \in \mathbb{Q}_{>0} \mid m^{\eta}_{ij} \leq u < m^{\xi}_{ij}\}$ and $I^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid M^{\eta}_{kl} < w \leq M^{\xi}_{kl}\}$ are non-empty. So it is possible to pick a value $t_{ij} = m^{\xi}_{ij} - \delta$ ($\delta \in \mathbb{Q}_{>0}$) in the interval $I^{\xi\Cup\eta}_{ij} \setminus I^{\xi}_{ij}$.
Next we must show that after collapsing $I^{\xi\Cup\eta}_{ij}$ of $\mathcal{D}^{\xi\Cup\eta}_{s}$ to t_{ij} (i.e., setting $m^{\xi\Cup\eta}_{ij} = M^{\xi\Cup\eta}_{ij} = t_{ij}$ and stabilizing $\mathcal{D}^{\xi\Cup\eta}_{s}$), hence obtaining $\boldsymbol{D}^{\xi\Cup\eta}_{s}$, it is still possible to pick a value t_{kl} in the interval $I^{\xi\Cup\eta}_{kl}$ of table $\boldsymbol{D}^{\xi\Cup\eta}_{s}$, such that t_{kl} belongs to I^{ξ}_{kl}, but does not belong to I^{η}_{kl}, and that $t_{ij} \leq t_{kl}$.
(We use bold font to distinguish items that are updated during stabilization.)
In $\mathcal{D}^{\xi\Cup\eta}_{s}$ we have $M^{\xi\Cup\eta}_{kj} \leq M^{\xi\Cup\eta}_{ki} + M^{\xi\Cup\eta}_{ij}$ and $M^{\xi\Cup\eta}_{kl} \leq M^{\xi\Cup\eta}_{kj} + M^{\xi\Cup\eta}_{jl}$.
Let $\mathcal{D} = \mathcal{D}^{\xi\Cup\eta}_{s}$. We set $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$ in interval $\boldsymbol{I}_{ij}$ of $\mathcal{D}$ and stabilise it to obtain $\boldsymbol{D}^{\xi\Cup\eta}_{s}$. After setting $\boldsymbol{M}^{\xi\Cup\eta}_{ij} = t_{ij}$ in $\mathcal{D}$, one of the following two things will happen:

(i) The inequation $M^{\xi\Cup\eta}_{kj} \leq M^{\xi\Cup\eta}_{ki} + t_{ij}$ holds, in which case, by Observation 7, $M^{\xi\Cup\eta}_{kj}$ will not change: $\boldsymbol{M}^{\xi\Cup\eta}_{kj} = M^{\xi\Cup\eta}_{kj}$. The inequation $M^{\xi\Cup\eta}_{kl} \leq \boldsymbol{M}^{\xi\Cup\eta}_{kj} + M^{\xi\Cup\eta}_{jl}$ will then continue to hold. Therefore $M^{\xi\Cup\eta}_{kl}$ will not change: $\boldsymbol{M}^{\xi\Cup\eta}_{kl} = M^{\xi\Cup\eta}_{kl} = M^{\xi}_{kl}$. Hence, $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid M^{\eta}_{kl} < w \leq M^{\xi}_{kl}\}$.

(ii) The inequation $M^{\xi\Cup\eta}_{kj} \leq M^{\xi\Cup\eta}_{ki} + t_{ij}$ does not hold. In that case, to restore the inequation, $M^{\xi\Cup\eta}_{kj}$ will be set to $M^{\xi\Cup\eta}_{ki} + t_{ij}$. By Observation 6, $M^{\xi\Cup\eta}_{kj}$ cannot further decrease, therefore $\boldsymbol{M}^{\xi\Cup\eta}_{kj} = M^{\xi\Cup\eta}_{ki} + m^{\xi}_{ij} - \delta$. But since $\boldsymbol{M}^{\xi\Cup\eta}_{kj}$ has changed, the inequation $M^{\xi\Cup\eta}_{kl} \leq \boldsymbol{M}^{\xi\Cup\eta}_{kj} + M^{\xi\Cup\eta}_{jl}$ might not hold. There are two cases to consider:
   1. If the new value of $\boldsymbol{M}^{\xi\Cup\eta}_{kj}$ does not affect the satisfiability of the inequation $M^{\xi\Cup\eta}_{kl} \leq \boldsymbol{M}^{\xi\Cup\eta}_{kj} + M^{\xi\Cup\eta}_{jl}$, then $M^{\xi\Cup\eta}_{kl}$ will remain the same and similar to case (i) $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid M^{\eta}_{kl} < w \leq M^{\xi}_{kl}\}$.
   2. If $M^{\xi\Cup\eta}_{kl} \leq \boldsymbol{M}^{\xi\Cup\eta}_{kj} + M^{\xi\Cup\eta}_{jl}$ is not satisfied, to restore the inequation, $M^{\xi\Cup\eta}_{kl}$ will be set to $\boldsymbol{M}^{\xi\Cup\eta}_{kj} + M^{\xi\Cup\eta}_{jl}$: $\boldsymbol{M}^{\xi\Cup\eta}_{kl} := \boldsymbol{M}^{\xi\Cup\eta}_{kj} + M^{\xi\Cup\eta}_{jl} = M^{\xi\Cup\eta}_{ki} + m^{\xi}_{ij} - \delta + M^{\xi\Cup\eta}_{jl}$.
   By the assumption, $M^{\xi\Cup\eta}_{ki} + m^{\xi}_{ij} + M^{\xi\Cup\eta}_{jl} > M^{\eta}_{kl}$. Therefore, $\boldsymbol{M}^{\xi\Cup\eta}_{kl} > m^{\eta}_{kl}$. So, $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid M^{\eta}_{kl} < w \leq \boldsymbol{M}^{\xi\Cup\eta}_{kl}\}$ will be non-empty.

If (i) or (ii).1, since $m^{\eta}_{ij} \leq m^{\xi}_{ij}$, $t_{ij} = m^{\xi}_{ij} - \delta$, and $m^{\xi}_{ij} \leq M^{\xi}_{ij}$ (the intervals I^{ξ}_{ij} and I^{η}_{ij} are overlapping), we have $t_{ij} \leq M^{\eta}_{ij} \leq M^{\eta}_{kl}$. So, any value that is picked for t_{kl} from $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl}$ will satisfy $t_{ij} \leq t_{kl}$.
If (ii).2, since $\boldsymbol{M}^{\xi\Cup\eta}_{kl} = M^{\xi\Cup\eta}_{ki} + t_{ij} + M^{\xi\Cup\eta}_{jl}$, we have $t_{ij} \leq \boldsymbol{M}^{\xi\Cup\eta}_{kl}$. Then it is possible to pick a value t_{kl} in $\boldsymbol{I}^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid M^{\eta}_{kl} < w \leq \boldsymbol{M}^{\xi\Cup\eta}_{kl}\}$ such that $t_{ij} \leq t_{kl}$.
Obviously, $t_{ij} \in \boldsymbol{I}^{\xi\Cup\eta}_{ij}$ and $t_{kl} \in \boldsymbol{I}^{\xi\Cup\eta}_{il}$. Moreover, $0 \leq k < i < j < l < n$, so, by Observation 3, it is possible to form a sequence $t_k \leq t_i \leq t_j \leq t_l$ that is compatible with $\boldsymbol{D}^{\xi\Cup\eta}_{s}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \Cup \eta]\!]$. Clearly, $t^{\mathcal{B}^z}_{ij} \notin I^{\xi}_{ij}$ and $t^{\mathcal{B}^z}_{kl} \notin I^{\eta}_{kl}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl.

(c) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \geq b$, $a < M^{\eta}_{ij}$, $m^{\xi}_{kl} < b$ and $m^{\xi\Cup\eta}_{ki} + a + m^{\xi\Cup\eta}_{jl} < b$.

Because of constraint α, $M^{\xi}_{ij} = a$ and because of constraint β, $m^{\eta}_{kl} = b$. $M^{\xi\Cup\eta}_{ij} = \max(M^{\xi}_{ij}, M^{\eta}_{ij}) = M^{\eta}_{ij}$ and $m^{\xi\Cup\eta}_{kl} = \min(m^{\xi}_{kl}, m^{\eta}_{kl}) = m^{\xi}_{kl}$.
Initially, both $I^{\xi\Cup\eta}_{ij} \setminus I^{\xi}_{ij} = \{u \in \mathbb{Q}_{>0} \mid M^{\xi}_{ij} < u \leq M^{\eta}_{ij}\}$ and $I^{\xi\Cup\eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{Q}_{\geq 0} \mid m^{\xi}_{kl} \leq w < m^{\eta}_{kl}\}$ are non-empty. So it is possible to pick a value $t_{ij} = M^{\xi}_{ij} + \delta$ ($\delta \in \mathbb{Q}_{>0}$) in the interval $I^{\xi\Cup\eta}_{ij} \setminus I^{\xi}_{ij}$.
Next we must show that after collapsing $I^{\xi\Cup\eta}_{ij}$ of $\mathcal{D}^{\xi\Cup\eta}_{s}$ to t_{ij} (i.e., setting $m^{\xi\Cup\eta}_{ij} = M^{\xi\Cup\eta}_{ij} = t_{ij}$ and stabilizing $\mathcal{D}^{\xi\Cup\eta}_{s}$), hence obtaining $\boldsymbol{D}^{\xi\Cup\eta}_{s}$, it is still

possible to pick a value t_{kl} in the interval $I_{kl}^{\xi \Cup \eta}$ of table $\boldsymbol{D}_s^{\xi \Cup \eta}$, such that t_{kl} belongs to I_{kl}^{ξ}, but does not belong to I_{kl}^{η}, and that $t_{ij} \leq t_{kl}$.
(We use bold font to distinguish items that are updated during stabilization.)
In $\mathcal{D}_s^{\xi \Cup \eta}$ we have $m_{ki}^{\xi \Cup \eta} + m_{ij}^{\xi \Cup \eta} \leq m_{kj}^{\xi \Cup \eta}$ and $m_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} \leq m_{kl}^{\xi \Cup \eta}$.
Let $\mathcal{D} = \mathcal{D}_s^{\xi \Cup \eta}$. We set $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$ in interval $\boldsymbol{I}_{ij}$ of $\mathcal{D}$ and stabilise it to obtain $\boldsymbol{D}_s^{\xi \Cup \eta}$. After setting $\boldsymbol{m}_{ij}^{\xi \Cup \eta} = t_{ij}$ in $\mathcal{D}$, one of the following two things will happen:

(i) The inequation $m_{ki}^{\xi \Cup \eta} + t_{ij} \leq m_{kj}^{\xi \Cup \eta}$ holds, in which case $m_{kj}^{\xi \Cup \eta}$ will not change: $\boldsymbol{m}_{kj}^{\xi \Cup \eta} = m_{kj}^{\xi \Cup \eta}$. Then, the inequation $m_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} \leq m_{kl}^{\xi \Cup \eta}$ continues to hold. Therefore, $m_{kl}^{\xi \Cup \eta}$ will not change: $\boldsymbol{m}_{kl}^{\xi \Cup \eta} = m_{kl}^{\xi \Cup \eta} = m_{kl}^{\xi}$. Hence, $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid m_{kl}^{\xi} \leq w < m_{kl}^{\eta}\}$.

(ii) The inequation $m_{ki}^{\xi \Cup \eta} + t_{ij} \leq m_{kj}^{\xi \Cup \eta}$ does not hold. In that case, to restore the inequation, $m_{kj}^{\xi \Cup \eta}$ will be set to $m_{ki}^{\xi \Cup \eta} + t_{ij}$: $\boldsymbol{m}_{kj}^{\xi \Cup \eta} := m_{ki}^{\xi \Cup \eta} + M_{ij}^{\xi} + \delta$. But since $\boldsymbol{m}_{kj}^{\xi \Cup \eta}$ has changed, the inequation $\boldsymbol{m}_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} \leq m_{kl}^{\xi \Cup \eta}$ might not hold. There are two cases to consider:
   1. If the new value of $\boldsymbol{m}_{kj}^{\xi \Cup \eta}$ does not affect the satisfiability of the inequation $\boldsymbol{m}_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} \leq m_{kl}^{\xi \Cup \eta}$, then $m_{kl}^{\xi \Cup \eta}$ will remain the same and similar to case (i) $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid m_{kl}^{\xi} \leq w < m_{kl}^{\eta}\}$.
   2. If $\boldsymbol{m}_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} \leq m_{kl}^{\xi \Cup \eta}$ does not hold, to restore the inequation, $m_{kl}^{\xi \Cup \eta}$ will be set to $\boldsymbol{m}_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta}$: $\boldsymbol{m}_{kl}^{\xi \Cup \eta} := \boldsymbol{m}_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} = m_{ki}^{\xi \Cup \eta} + M_{ij}^{\xi} + \delta + m_{jl}^{\xi \Cup \eta}$.
   By the assumption, $m_{ki}^{\xi \Cup \eta} + M_{ij}^{\xi} + m_{jl}^{\xi \Cup \eta} < m_{kl}^{\eta}$. Therefore, it is always possible to pick a value for δ so that $\boldsymbol{m}_{kl}^{\xi \Cup \eta} < m_{kl}^{\eta}$. So, $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid \boldsymbol{m}_{kl}^{\xi \Cup \eta} \leq w < m_{kl}^{\eta}\}$ will be non-empty.

If (i) or (ii).1, since $m_{kl}^{\xi} = m_{kl}^{\xi \Cup \eta} \leq \boldsymbol{m}_{kl}^{\xi \Cup \eta}$, then any value t_{kl} in $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid m_{kl}^{\xi} \leq w < m_{kl}^{\eta}\}$ such that $t_{kl} > m_{ki}^{\xi \Cup \eta} + t_{ij} + m_{jl}^{\xi \Cup \eta}$ will satisfy $t_{ij} \leq t_{kl}$.
If (ii).2, since $\boldsymbol{m}_{kl}^{\xi \Cup \eta} = m_{ki}^{\xi \Cup \eta} + t_{ij} + m_{jl}^{\xi \Cup \eta}$, we have $t_{ij} \leq \boldsymbol{m}_{kl}^{\xi \Cup \eta}$. Then any value t_{kl} in $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid \boldsymbol{m}_{kl}^{\xi \Cup \eta} \leq w < m_{kl}^{\eta}\}$ will satisfy $t_{ij} \leq t_{kl}$.
Obviously, $t_{ij} \in \boldsymbol{I}_{ij}^{\xi \Cup \eta}$ and $t_{kl} \in \boldsymbol{I}_{il}^{\xi \Cup \eta}$. Moreover, $0 \leq k < i < j < l < n$, so, by Observation 3, it is possible to form a sequence $t_k \leq t_i \leq t_j \leq t_l$ that is compatible with $\boldsymbol{D}_s^{\xi \Cup \eta}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \Cup \eta]\!]$. Clearly, $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$ and $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl.

(d) $\underline{\alpha = \tau_{i,j} \leq a,\ \beta = \tau_{k,l} \leq b,\ a < M_{ij}^{\eta},\ b < m_{kl}^{\xi}.}$

Because of constraint α, $M_{ij}^{\xi} = a$ and because of constraint β, $M_{kl}^{\eta} = b$. $M_{ij}^{\xi \Cup \eta} = \max(M_{ij}^{\xi}, M_{ij}^{\eta}) = M_{ij}^{\eta}$ and $M_{kl}^{\xi \Cup \eta} = \max(M_{kl}^{\xi}, M_{kl}^{\eta}) = M_{kl}^{\xi}$.

Initially, both $I_{ij}^{\xi\Cup\eta} \setminus I_{ij}^{\xi} = \{u \in \mathbb{Q}_{>0} \mid M_{ij}^{\xi} < u \leq M_{ij}^{\eta}\}$ and $I_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid M_{kl}^{\eta} < w \leq M_{kl}^{\xi}\}$ are non-empty. So it is possible to pick a value $t_{ij} = M_{ij}^{\xi} + \delta$ ($\delta \in \mathbb{Q}_{>0}$) in the interval $I_{ij}^{\xi\Cup\eta} \setminus I_{ij}^{\xi}$.
Next we must show that after collapsing $I_{ij}^{\xi\Cup\eta}$ of $\mathcal{D}_s^{\xi\Cup\eta}$ to t_{ij} (i.e., setting $m_{ij}^{\xi\Cup\eta} = M_{ij}^{\xi\Cup\eta} = t_{ij}$ and stabilizing $\mathcal{D}_s^{\xi\Cup\eta}$), hence obtaining $\boldsymbol{D}_s^{\xi\Cup\eta}$, it is still possible to pick a value t_{kl} in the interval $I_{kl}^{\xi\Cup\eta}$ of table $\boldsymbol{D}_s^{\xi\Cup\eta}$, such that t_{kl} belongs to I_{kl}^{ξ}, but does not belong to I_{kl}^{η}, and that $t_{ij} \leq t_{kl}$.
(We use bold font to distinguish items that are updated during stabilization.)
In $\mathcal{D}_s^{\xi\Cup\eta}$ we have $M_{kj}^{\xi\Cup\eta} \leq M_{ki}^{\xi\Cup\eta} + M_{ij}^{\xi\Cup\eta}$ and $M_{kl}^{\xi\Cup\eta} \leq M_{kj}^{\xi\Cup\eta} + M_{jl}^{\xi\Cup\eta}$.
Let $\mathcal{D} = \mathcal{D}_s^{\xi\Cup\eta}$. We set $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$ in interval $\boldsymbol{I}_{ij}$ of $\mathcal{D}$ and stabilise it to obtain $\boldsymbol{D}_s^{\xi\Cup\eta}$. After setting $\boldsymbol{M}_{ij}^{\xi\Cup\eta} = t_{ij}$ in $\mathcal{D}$, one of the following two things will happen:

(i) The inequation $M_{kj}^{\xi\Cup\eta} \leq M_{ki}^{\xi\Cup\eta} + t_{ij}$ holds, in which case $M_{kj}^{\xi\Cup\eta}$ will not change: $\boldsymbol{M}_{kj}^{\xi\Cup\eta} = M_{kj}^{\xi\Cup\eta}$. Then, the inequation $M_{kl}^{\xi\Cup\eta} \leq \boldsymbol{M}_{kj}^{\xi\Cup\eta} + M_{jl}^{\xi\Cup\eta}$ continues to hold. Therefore, $\boldsymbol{M}_{kl}^{\xi\Cup\eta} = M_{kl}^{\xi\Cup\eta} = M_{kl}^{\xi}$. Hence, $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid M_{kl}^{\eta} < w \leq M_{kl}^{\xi}\}$.

(ii) The inequation $M_{kj}^{\xi\Cup\eta} \leq M_{ki}^{\xi\Cup\eta} + t_{ij}$ does not hold. In that case, to restore the inequation, $M_{kj}^{\xi\Cup\eta}$ will be set to $M_{ki}^{\xi\Cup\eta} + t_{ij}$: $\boldsymbol{M}_{kj}^{\xi\Cup\eta} := M_{ki}^{\xi\Cup\eta} + M_{ij}^{\xi} + \delta$. But since $\boldsymbol{M}_{kj}^{\xi\Cup\eta}$ has changed, the inequation $M_{kl}^{\xi\Cup\eta} \leq \boldsymbol{M}_{kj}^{\xi\Cup\eta} + M_{jl}^{\xi\Cup\eta}$ might not hold. There are two cases to consider:
   1. If the new value of $\boldsymbol{M}_{kj}^{\xi\Cup\eta}$ does not affect the satisfiability of the inequation, then $M_{kl}^{\xi\Cup\eta}$ will remain the same and similar to case (i) $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid M_{kl}^{\eta} < w \leq M_{kl}^{\xi}\}$.
   2. If $M_{kl}^{\xi\Cup\eta} \leq \boldsymbol{M}_{kj}^{\xi\Cup\eta} + M_{jl}^{\xi\Cup\eta}$ does not hold, to restore the inequation, $M_{kl}^{\xi\Cup\eta}$ will be set to $\boldsymbol{M}_{kj}^{\xi\Cup\eta} + M_{jl}^{\xi\Cup\eta}$: $\boldsymbol{M}_{kl}^{\xi\Cup\eta} := \boldsymbol{M}_{kj}^{\xi\Cup\eta} + M_{jl}^{\xi\Cup\eta} = M_{ki}^{\xi\Cup\eta} + M_{ij}^{\xi} + \delta + M_{jl}^{\xi\Cup\eta}$. Next, we show that $M_{ki}^{\xi\Cup\eta} + M_{ij}^{\xi} + M_{jl}^{\xi\Cup\eta} > M_{kl}^{\eta}$.
   Assume $M_{ki}^{\xi\Cup\eta} + M_{ij}^{\xi} + M_{jl}^{\xi\Cup\eta} \leq M_{kl}^{\eta}$. Because $M_{kl}^{\eta} < M_{kl}^{\xi}$, we will have $M_{ki}^{\xi\Cup\eta} + M_{ij}^{\xi} + M_{jl}^{\xi\Cup\eta} < M_{kl}^{\xi}$.
   But $M_{ki}^{\xi} \leq M_{ki}^{\xi\Cup\eta}$ and $M_{jl}^{\xi} \leq M_{jl}^{\xi\Cup\eta}$, so $M_{ki}^{\xi} + M_{ij}^{\xi} + M_{jl}^{\xi} < M_{kl}^{\xi}$.
   Because $M_{kj}^{\xi} \leq M_{ki}^{\xi} + M_{ij}^{\xi}$, we have $M_{kj}^{\xi} + M_{jl}^{\xi} < M_{kl}^{\xi}$. Because $M_{kl}^{\xi} \leq M_{kj}^{\xi} + M_{jl}^{\xi}$, we have $M_{kl}^{\xi} < M_{kl}^{\xi}$: a contradiction.
   Because $M_{ki}^{\xi\Cup\eta} + M_{ij}^{\xi} + M_{jl}^{\xi\Cup\eta} > M_{kl}^{\eta}$, we will have $\boldsymbol{M}_{kl}^{\xi\Cup\eta} > M_{kl}^{\eta}$.
   So $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid M_{kl}^{\eta} < w \leq \boldsymbol{M}_{kl}^{\xi\Cup\eta}\}$ will be non-empty.

If (i) or (ii).1, $t_{ij} = M_{ij}^{\xi} + \delta \leq M_{ij}^{\eta}$. But because $M_{ij}^{\eta} \leq M_{kl}^{\eta}$, we have $t_{ij} \leq M_{kl}^{\eta}$. Therefore, any value that is picked for t_{kl} from $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta}$ will satisfy $t_{ij} \leq t_{kl}$.
If (ii).2, since $\boldsymbol{M}_{kl}^{\xi\Cup\eta} = M_{ki}^{\xi\Cup\eta} + t_{ij} + M_{jl}^{\xi\Cup\eta}$, we have $t_{ij} \leq \boldsymbol{M}_{kl}^{\xi\Cup\eta}$. Then it is possible to pick a value t_{kl} in $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid M_{kl}^{\eta} < w \leq \boldsymbol{M}_{kl}^{\xi\Cup\eta}\}$ such that $t_{ij} \leq t_{kl}$.

Obviously, $t_{ij} \in \boldsymbol{I}_{ij}^{\xi \uplus \eta}$ and $t_{kl} \in \boldsymbol{I}_{il}^{\xi \uplus \eta}$. Moreover, $0 \leq k < i < j < l < n$, so, by Observation 3, it is possible to form a sequence $t_k \leq t_i \leq t_j \leq t_l$ that is compatible with $\boldsymbol{D}_s^{\xi \uplus \eta}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \uplus \eta]\!]$. Clearly, $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$ and $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl.

case 2: $0 \leq i < k < j < l < n$.

(a) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \geq b$, $m_{ij}^{\eta} < a$, $m_{kl}^{\xi} < b$ and $m_{il}^{\xi \uplus \eta} - a < b - m_{kj}^{\xi \uplus \eta}$

Because of constraint α, $m_{ij}^{\xi} = a$ and because of constraint β, $m_{kl}^{\eta} = b$. $m_{ij}^{\xi \uplus \eta} = \min(m_{ij}^{\xi}, m_{ij}^{\eta}) = m_{ij}^{\eta}$ and $m_{kl}^{\xi \uplus \eta} = \min(m_{kl}^{\xi}, m_{kl}^{\eta}) = m_{kl}^{\xi}$.
Initially, both $I_{ij}^{\xi \uplus \eta} \setminus I_{ij}^{\xi} = \{u \in \mathbb{R}_{>0} \mid m_{ij}^{\eta} \leq u < m_{ij}^{\xi}\}$ and $I_{kl}^{\xi \uplus \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid m_{kl}^{\xi} \leq u < m_{kl}^{\eta}\}$ are non-empty. So it is possible to pick a value $t_{ij} = m_{ij}^{\xi} - \delta$ ($\delta \in \mathbb{R}_{>0}$) in the interval $I_{ij}^{\xi \uplus \eta} \setminus I_{ij}^{\xi}$.
Next we must show that after collapsing $I_{ij}^{\xi \uplus \eta}$ to t_{ij} (i.e., setting $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$ in interval $I_{ij}^{\xi \uplus \eta}$ of $\mathcal{D}_s^{\xi \uplus \eta}$ and stabilizing the table), it is still possible to pick a value $t_{kl} \in \boldsymbol{I}_{kl}^{\xi \uplus \eta}$ of table $\boldsymbol{D}_s^{\xi \uplus \eta}$, such that t_{kl} belongs to I_{kl}^{ξ}, but does not belong to I_{kl}^{η}. In other words, we must show $\boldsymbol{I}_{kl}^{\xi \uplus \eta} \setminus I_{kl}^{\eta} \neq \emptyset$.
In $\mathcal{D}_s^{\xi \uplus \eta}$ we have $M_{ik}^{\xi \uplus \eta} + m_{kj}^{\xi \uplus \eta} \leq M_{ij}^{\xi \uplus \eta}$. After setting $\boldsymbol{M}_{ij}^{\xi \uplus \eta} = t_{ij}$, one of the following two things will happen:

(i) The inequation continues to hold, in which case $M_{ik}^{\xi \uplus \eta}$ will not change.
(ii) The inequation does no longer hold. In that case, to restore the inequation, $M_{ik}^{\xi \uplus \eta}$ will be set to $t_{ij} - m_{kj}^{\xi \uplus \eta}$: $\boldsymbol{M}_{ik}^{\xi \uplus \eta} := m_{ij}^{\xi} - \delta - m_{kj}^{\xi \uplus \eta}$.

If (i), then the inequation $m_{il}^{\xi \uplus \eta} \leq M_{ik}^{\xi \uplus \eta} + m_{kl}^{\xi \uplus \eta}$ continues to hold and therefore, $m_{kl}^{\xi \uplus \eta}$ will not change: $\boldsymbol{m}_{kl}^{\xi \uplus \eta} = m_{kl}^{\xi \uplus \eta}$. Therefore, $\boldsymbol{I}_{kl}^{\xi \uplus \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid m_{kl}^{\xi} \leq u < m_{kl}^{\eta}\}$.
If (ii), then since $\boldsymbol{M}_{ik}^{\xi \uplus \eta}$ has changed, the inequation $m_{il}^{\xi \uplus \eta} \leq \boldsymbol{M}_{ik}^{\xi \uplus \eta} + m_{kl}^{\xi \uplus \eta}$ might no longer hold. There are two cases to consider:

– If the new value of $\boldsymbol{M}_{ik}^{\xi \uplus \eta}$ does not affect the satisfiability of the inequation, then $m_{kl}^{\xi \uplus \eta}$ will remain the same and similar to case (i) $\boldsymbol{I}_{kl}^{\xi \uplus \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid m_{kl}^{\xi} \leq u < m_{kl}^{\eta}\}$.
– If $m_{il}^{\xi \uplus \eta} \leq \boldsymbol{M}_{ik}^{\xi \uplus \eta} + m_{kl}^{\xi \uplus \eta}$ is not satisfied, to restore the inequation, $\boldsymbol{m}_{kl}^{\xi \uplus \eta}$ will be set to $m_{il}^{\xi \uplus \eta} - \boldsymbol{M}_{ik}^{\xi \uplus \eta} = m_{il}^{\xi \uplus \eta} - (m_{ij}^{\xi} - \delta - m_{kj}^{\xi \uplus \eta}) = m_{il}^{\xi \uplus \eta} - m_{ij}^{\xi} + \delta + m_{kj}^{\xi \uplus \eta}$.
  By the assumption, $m_{il}^{\xi \uplus \eta} - m_{ij}^{\xi} < m_{kl}^{\eta} - m_{kj}^{\xi \uplus \eta}$, so $m_{il}^{\xi \uplus \eta} - m_{ij}^{\xi} + m_{kj}^{\xi \uplus \eta} < m_{kl}^{\eta}$. So it is always possible to choose an appropriate value for δ so that $\boldsymbol{m}_{kl}^{\xi \uplus \eta} < m_{kl}^{\eta}$. So $\boldsymbol{I}_{kl}^{\xi \uplus \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid \boldsymbol{m}_{kl}^{\xi \uplus \eta} \leq u < m_{kl}^{\eta}\}$ will be non-empty.

So in both cases (i) and (ii) it is always possible to pick a value for t_{kl} from $\boldsymbol{I}_{kl}^{\xi \uplus \eta} \setminus I_{kl}^{\eta}$. Obviously, $t_{ij} \in \boldsymbol{I}_{ij}^{\xi \uplus \eta}$ and $t_{kl} \in \boldsymbol{I}_{il}^{\xi \uplus \eta}$. Moreover, $0 \leq i < k < j < l < n$, so, by Observation 3, it is possible to form a sequence

$t_i \leq t_k \leq t_j \leq t_l$ that is compatible with $\boldsymbol{D}_s^{\xi \Cup \eta}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \Cup \eta]\!]$. Clearly, $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$ and $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl.

(b) $\underline{\alpha = \tau_{i,j} \geq a,\ \beta = \tau_{k,l} \leq b,\ m_{ij}^{\eta} < a,\ b < M_{kl}^{\xi} \text{ and } a + M_{jl}^{\xi \Cup \eta} > m_{ik}^{\xi \Cup \eta} + b.}$

Because of constraint α, $m_{ij}^{\xi} = a$ and because of constraint β, $M_{kl}^{\eta} = b$. $m_{ij}^{\xi \Cup \eta} = \min(m_{ij}^{\xi}, m_{ij}^{\eta}) = m_{ij}^{\eta}$ and $M_{kl}^{\xi \Cup \eta} = \max(M_{kl}^{\xi}, M_{kl}^{\eta}) = M_{kl}^{\xi}$.

Initially, both $I_{ij}^{\xi \Cup \eta} \setminus I_{ij}^{\xi} = \{u \in \mathbb{R}_{>0} \mid m_{ij}^{\eta} \leq u < m_{ij}^{\xi}\}$ and $I_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid M_{kl}^{\eta} < u \leq M_{kl}^{\xi}\}$ are non-empty. So it is possible to pick a value $t_{ij} = m_{ij}^{\xi} - \delta$ ($\delta \in \mathbb{R}_{>0}$) in the interval $I_{ij}^{\xi \Cup \eta} \setminus I_{ij}^{\xi}$.

Next we must show that after collapsing $I_{ij}^{\xi \Cup \eta}$ to t_{ij} (i.e., setting $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$ in interval $I_{ij}^{\xi \Cup \eta}$ of $\mathcal{D}_s^{\xi \Cup \eta}$ and stabilizing the table), it is still possible to pick a value $t_{kl} \in \boldsymbol{I}_{kl}^{\xi \Cup \eta}$ of table $\boldsymbol{D}_s^{\xi \Cup \eta}$, such that t_{kl} belongs to I_{kl}^{ξ}, but does not belong to I_{kl}^{η}. In other words, we must show $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} \neq \emptyset$. In $\mathcal{D}_s^{\xi \Cup \eta}$ we have $m_{ik}^{\xi \Cup \eta} + M_{kj}^{\xi \Cup \eta} \leq M_{ij}^{\xi \Cup \eta}$. After setting $\boldsymbol{M}_{ij}^{\xi \Cup \eta} = t_{ij}$, one of the following two things will happen:

(i) The inequation continues to hold, in which case $M_{kj}^{\xi \Cup \eta}$ will not change.
(ii) The inequation does no longer hold. In that case, to restore the inequation, $M_{kj}^{\xi \Cup \eta}$ will be set to $t_{ij} - m_{ik}^{\xi \Cup \eta}$: $\boldsymbol{M}_{kj}^{\xi \Cup \eta} := m_{ij}^{\xi} - \delta - m_{ik}^{\xi \Cup \eta}$.

If (i), then the inequation $M_{kl}^{\xi \Cup \eta} \leq M_{kj}^{\xi \Cup \eta} + M_{jl}^{\xi \Cup \eta}$ continues to hold and therefore, $M_{kl}^{\xi \Cup \eta}$ will not change: $\boldsymbol{M}_{kl}^{\xi \Cup \eta} = M_{kl}^{\xi \Cup \eta}$. Therefore, $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid M_{kl}^{\eta} < u \leq M_{kl}^{\xi}\}$.

If (ii), then since $\boldsymbol{M}_{kj}^{\xi \Cup \eta}$ has changed, the inequation $M_{kl}^{\xi \Cup \eta} \leq \boldsymbol{M}_{kj}^{\xi \Cup \eta} + M_{jl}^{\xi \Cup \eta}$ might no longer hold. There are two cases to consider:

– If the new value of $\boldsymbol{M}_{kj}^{\xi \Cup \eta}$ does not affect the satisfiability of the inequation, then $M_{kl}^{\xi \Cup \eta}$ will remain the same and similar to case (i) $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid M_{kl}^{\eta} < u \leq M_{kl}^{\xi}\}$.
– If $M_{kl}^{\xi \Cup \eta} \leq \boldsymbol{M}_{kj}^{\xi \Cup \eta} + M_{jl}^{\xi \Cup \eta}$ is not satisfied, to restore the inequation, $\boldsymbol{M}_{kl}^{\xi \Cup \eta}$ will be set to $\boldsymbol{M}_{kj}^{\xi \Cup \eta} + M_{jl}^{\xi \Cup \eta} = m_{ij}^{\xi} - \delta - m_{ik}^{\xi \Cup \eta} + M_{jl}^{\xi \Cup \eta}$.
  By the assumption, $m_{ij}^{\xi} + M_{jl}^{\xi \Cup \eta} > m_{ik}^{\xi \Cup \eta} + M_{kl}^{\eta}$, so $m_{ij}^{\xi} - m_{ik}^{\xi \Cup \eta} + M_{jl}^{\xi \Cup \eta} > M_{kl}^{\eta}$. So it is always possible to choose an appropriate value for δ so that $\boldsymbol{M}_{kl}^{\xi \Cup \eta} > M_{kl}^{\eta}$. So $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid M_{kl}^{\eta} < u \leq \boldsymbol{M}_{kl}^{\xi \Cup \eta}\}$ will be non-empty.

So in both cases (i) and (ii) it is always possible to pick a value for t_{kl} from $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta}$. Obviously, $t_{ij} \in \boldsymbol{I}_{ij}^{\xi \Cup \eta}$ and $t_{kl} \in \boldsymbol{I}_{il}^{\xi \Cup \eta}$. Moreover, $0 \leq i < k < j < l < n$, so, by Observation 3, it is possible to form a sequence $t_i \leq t_k \leq t_j \leq t_l$ that is compatible with $\boldsymbol{D}_s^{\xi \Cup \eta}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \Cup \eta]\!]$. Clearly, $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$ and $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl.

(c) $\underline{\alpha = \tau_{i,j} \leq a,\ \beta = \tau_{k,l} \geq b,\ a < M_{ij}^{\eta},\ m_{kl}^{\xi} < b \text{ and } a + m_{jl}^{\xi \Cup \eta} < M_{ik}^{\xi \Cup \eta} + b.}$

Because of constraint α, $M_{ij}^{\xi} = a$ and because of constraint β, $m_{kl}^{\eta} = b$. $M_{ij}^{\xi \Cup \eta} = \max(M_{ij}^{\xi}, M_{ij}^{\eta}) = M_{ij}^{\eta}$ and $m_{kl}^{\xi \Cup \eta} = \min(m_{kl}^{\xi}, m_{kl}^{\eta}) = m_{kl}^{\xi}$.
Initially, both $I_{ij}^{\xi \Cup \eta} \setminus I_{ij}^{\xi} = \{u \in \mathbb{R}_{>0} \mid M_{ij}^{\xi} < u \leq M_{ij}^{\eta}\}$ and $I_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid m_{kl}^{\xi} \leq u < m_{kl}^{\eta}\}$ are non-empty. So it is possible to pick a value $t_{ij} = M_{ij}^{\xi} + \delta$ ($\delta \in \mathbb{R}_{>0}$) in the interval $I_{ij}^{\xi \Cup \eta} \setminus I_{ij}^{\xi}$.
Next we must show that after collapsing $I_{ij}^{\xi \Cup \eta}$ to t_{ij} (i.e., setting $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$ in interval $I_{ij}^{\xi \Cup \eta}$ of $\mathcal{D}_{s}^{\xi \Cup \eta}$ and stabilizing the table), it is still possible to pick a value $t_{kl} \in \boldsymbol{I}_{kl}^{\xi \Cup \eta}$ of table $\boldsymbol{D}_{s}^{\xi \Cup \eta}$, such that t_{kl} belongs to I_{kl}^{ξ}, but does not belong to I_{kl}^{η}. In other words, we must show $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} \neq \emptyset$.
In $\mathcal{D}_{s}^{\xi \Cup \eta}$ we have $m_{ij}^{\xi \Cup \eta} \leq M_{ik}^{\xi \Cup \eta} + m_{kj}^{\xi \Cup \eta}$. After setting $\boldsymbol{m}_{ij}^{\xi \Cup \eta} = t_{ij}$, one of the following two things will happen:

(i) The inequation continues to hold, in which case $m_{kj}^{\xi \Cup \eta}$ will not change.
(ii) The inequation does no longer hold. In that case, to restore the inequation, $m_{kj}^{\xi \Cup \eta}$ will be set to $t_{ij} - M_{ik}^{\xi \Cup \eta}$: $\boldsymbol{m}_{kj}^{\xi \Cup \eta} := M_{ij}^{\xi} + \delta - M_{ik}^{\xi \Cup \eta}$.

If (i), then $m_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} \leq m_{kl}^{\xi \Cup \eta}$ continues to hold and therefore, $m_{kl}^{\xi \Cup \eta}$ will not change: $\boldsymbol{m}_{kl}^{\xi \Cup \eta} = m_{kl}^{\xi \Cup \eta}$. Hence, $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid m_{kl}^{\xi} \leq u < m_{kl}^{\eta}\}$.
If (ii), then since $\boldsymbol{m}_{kj}^{\xi \Cup \eta}$ has changed, the inequation $\boldsymbol{m}_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} \leq m_{kl}^{\xi \Cup \eta}$ might no longer hold. There are two cases to consider:

– If the new value of $\boldsymbol{m}_{kj}^{\xi \Cup \eta}$ does not affect the satisfiability of the inequation, then $m_{kl}^{\xi \Cup \eta}$ will remain the same and similar to case (i) $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid m_{kl}^{\xi} \leq u < m_{kl}^{\eta}\}$.
– If $\boldsymbol{m}_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} \leq m_{kl}^{\xi \Cup \eta}$ is not satisfied, to restore the inequation, $\boldsymbol{m}_{kl}^{\xi \Cup \eta}$ will be set to $\boldsymbol{m}_{kj}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} = M_{ij}^{\xi} + \delta - M_{ik}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta}$.
By the assumption, $M_{ij}^{\xi} + m_{jl}^{\xi \Cup \eta} < M_{ik}^{\xi \Cup \eta} + m_{kl}^{\eta}$, so $M_{ij}^{\xi} - M_{ik}^{\xi \Cup \eta} + m_{jl}^{\xi \Cup \eta} < m_{kl}^{\eta}$. So it is always possible to choose an appropriate value for δ so that $\boldsymbol{m}_{kl}^{\xi \Cup \eta} < m_{kl}^{\eta}$. So $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid \boldsymbol{m}_{kl}^{\xi \Cup \eta} \leq u < m_{kl}^{\eta}\}$ will be non-empty.

So in both cases (i) and (ii) it is always possible to pick a value for t_{kl} from $\boldsymbol{I}_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta}$. Obviously, $t_{ij} \in \boldsymbol{I}_{ij}^{\xi \Cup \eta}$ and $t_{kl} \in \boldsymbol{I}_{il}^{\xi \Cup \eta}$. Moreover, $0 \leq i < k < j < l < n$, so, by Observation 3, it is possible to form a sequence $t_i \leq t_k \leq t_j \leq t_l$ that is compatible with $\boldsymbol{D}_{s}^{\xi \Cup \eta}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \Cup \eta]\!]$. Clearly, $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$ and $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl.

(d) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \leq b$, $a < M_{ij}^{\eta}$, $b < m_{kl}^{\xi}$ and $M_{il}^{\xi \Cup \eta} - a > b - M_{kj}^{\xi \Cup \eta}$.

Because of constraint α, $M_{ij}^{\xi} = a$ and because of constraint β, $M_{kl}^{\eta} = b$. $M_{ij}^{\xi \Cup \eta} = \max(M_{ij}^{\xi}, M_{ij}^{\eta}) = M_{ij}^{\eta}$ and $M_{kl}^{\xi \Cup \eta} = \max(M_{kl}^{\xi}, M_{kl}^{\eta}) = M_{kl}^{\xi}$.
Initially, both $I_{ij}^{\xi \Cup \eta} \setminus I_{ij}^{\xi} = \{u \in \mathbb{R}_{>0} \mid M_{ij}^{\xi} < u \leq M_{ij}^{\eta}\}$ and $I_{kl}^{\xi \Cup \eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid M_{kl}^{\eta} < u \leq M_{kl}^{\xi}\}$ are non-empty. So it is possible to pick a value $t_{ij} = M_{ij}^{\xi} + \delta$ ($\delta \in \mathbb{R}_{>0}$) in the interval $I_{ij}^{\xi \Cup \eta} \setminus I_{ij}^{\xi}$.

Next we must show that after collapsing $I_{ij}^{\xi\Cup\eta}$ to t_{ij} (i.e., setting $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$ in interval $I_{ij}^{\xi\Cup\eta}$ of $\mathcal{D}_s^{\xi\Cup\eta}$ and stabilizing the table), it is still possible to pick a value $t_{kl} \in \boldsymbol{I}_{kl}^{\xi\Cup\eta}$ of table $\boldsymbol{D}_s^{\xi\Cup\eta}$, such that t_{kl} belongs to I_{kl}^{ξ}, but does not belong to I_{kl}^{η}. In other words, we must show $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} \neq \emptyset$. In $\mathcal{D}_s^{\xi\Cup\eta}$ we have $m_{ij}^{\xi\Cup\eta} \leq m_{ik}^{\xi\Cup\eta} + M_{kj}^{\xi\Cup\eta}$. After setting $\boldsymbol{m}_{ij}^{\xi\Cup\eta} = t_{ij}$, one of the following two things will happen:

(i) The inequation continues to hold, in which case $m_{ik}^{\xi\Cup\eta}$ will not change.
(ii) The inequation does no longer hold. In that case, to restore the inequation, $m_{ik}^{\xi\Cup\eta}$ will be set to $t_{ij} - M_{kj}^{\xi\Cup\eta}$: $\boldsymbol{m}_{ik}^{\xi\Cup\eta} := M_{ij}^{\xi} + \delta - M_{kj}^{\xi\Cup\eta}$.

If (i), then the inequation $m_{ik}^{\xi\Cup\eta} + M_{kl}^{\xi\Cup\eta} \leq M_{il}^{\xi\Cup\eta}$ continues to hold and therefore, $M_{kl}^{\xi\Cup\eta}$ will not change: $\boldsymbol{M}_{kl}^{\xi\Cup\eta} = M_{kl}^{\xi\Cup\eta}$. Therefore, $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid M_{kl}^{\eta} < u \leq M_{kl}^{\xi}\}$.
If (ii), then since $\boldsymbol{m}_{ik}^{\xi\Cup\eta}$ has changed, the inequation $\boldsymbol{m}_{ik}^{\xi\Cup\eta} + M_{kl}^{\xi\Cup\eta} \leq M_{il}^{\xi\Cup\eta}$ might no longer hold. There are two cases to consider:

– If the new value of $\boldsymbol{m}_{ik}^{\xi\Cup\eta}$ does not affect the satisfiability of the inequation, then $\boldsymbol{M}_{kl}^{\xi\Cup\eta}$ will remain the same and similar to case (i) $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid M_{kl}^{\eta} < u \leq M_{kl}^{\xi}\}$.
– If $\boldsymbol{m}_{ik}^{\xi\Cup\eta} + M_{kl}^{\xi\Cup\eta} \leq M_{il}^{\xi\Cup\eta}$ is not satisfied, to restore the inequation, $\boldsymbol{M}_{kl}^{\xi\Cup\eta}$ will be set to $M_{il}^{\xi\Cup\eta} - \boldsymbol{m}_{ik}^{\xi\Cup\eta} = M_{il}^{\xi\Cup\eta} - (M_{ij}^{\xi} + \delta - M_{kj}^{\xi\Cup\eta}) = M_{il}^{\xi\Cup\eta} - M_{ij}^{\xi} - \delta + M_{kj}^{\xi\Cup\eta}$.
  By the assumption, $M_{il}^{\xi\Cup\eta} - M_{ij}^{\xi} > M_{kl}^{\eta} - M_{kj}^{\xi\Cup\eta}$, so $M_{il}^{\xi\Cup\eta} - M_{ij}^{\xi} + M_{kj}^{\xi\Cup\eta} > M_{kl}^{\eta}$. So it is always possible to choose an appropriate value for δ so that $\boldsymbol{M}_{kl}^{\xi\Cup\eta} > M_{kl}^{\eta}$. So $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} = \{u \in \mathbb{R}_{\geq 0} \mid M_{kl}^{\eta} < u \leq \boldsymbol{M}_{kl}^{\xi\Cup\eta}\}$ will be non-empty.

So in both cases (i) and (ii) it is always possible to pick a value for t_{kl} from $\boldsymbol{I}_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta}$. Obviously, $t_{ij} \in \boldsymbol{I}_{ij}^{\xi\Cup\eta}$ and $t_{kl} \in \boldsymbol{I}_{il}^{\xi\Cup\eta}$. Moreover, $0 \leq i < k < j < l < n$, so, by Observation 3, it is possible to form a sequence $t_i \leq t_k \leq t_j \leq t_l$ that is compatible with $\boldsymbol{D}_s^{\xi\Cup\eta}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \Cup \eta]\!]$. Clearly, $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$ and $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl.

<u>case 3 $0 \leq i < j \leq k < l < n$.</u>

(a) <u>$\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \geq b$, $m_{ij}^{\eta} < a$, $m_{kl}^{\xi} < b$</u>

Because of constraint α, $m_{ij}^{\xi} = a$ and because of constraint β, $m_{kl}^{\eta} = b$. $m_{ij}^{\xi\Cup\eta} = \min(m_{ij}^{\xi}, m_{ij}^{\eta}) = m_{ij}^{\eta}$ and $m_{kl}^{\xi\Cup\eta} = \min(m_{kl}^{\xi}, m_{kl}^{\eta}) = m_{kl}^{\xi}$.
$I_{ij}^{\xi\Cup\eta} \setminus I_{ij}^{\xi} = \{u \in \mathbb{R}_{\geq 0} \mid m_{ij}^{\eta} \leq u < m_{ij}^{\xi}\}$ is non-empty. So it is possible to pick a value $t_{ij} = m_{ij}^{\xi} - \delta$ ($\delta \in \mathbb{R}_{>0}$) in the interval $I_{ij}^{\xi\Cup\eta} \setminus I_{ij}^{\xi}$.
$I_{kl}^{\xi\Cup\eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{R}_{\geq 0} \mid m_{kl}^{\xi} \leq w < m_{kl}^{\eta}\}$ is non-empty.

(b) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \leq b$, $m^{\eta}_{ij} < a$, $b < M^{\xi}_{kl}$

Because of constraint α, $m^{\xi}_{ij} = a$ and because of constraint β, $M^{\eta}_{kl} = b$. $m^{\xi \Cup \eta}_{ij} = \min(m^{\xi}_{ij}, m^{\eta}_{ij}) = m^{\eta}_{ij}$ and $M^{\xi \Cup \eta}_{kl} = \max(M^{\xi}_{kl}, M^{\eta}_{kl}) = M^{\xi}_{kl}$. $I^{\xi \Cup \eta}_{ij} \setminus I^{\xi}_{ij} = \{u \in \mathbb{R}_{\geq 0} \mid m^{\eta}_{ij} \leq u < m^{\xi}_{ij}\}$ is non-empty. So it is possible to pick a value $t_{ij} = m^{\xi}_{ij} - \delta$ ($\delta \in \mathbb{R}_{>0}$) in the interval $I^{\xi \Cup \eta}_{ij} \setminus I^{\xi}_{ij}$. $I^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{R}_{\geq 0} \mid M^{\eta}_{kl} < w \leq M^{\xi}_{kl}\}$ is non-empty.

(c) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \geq b$, $a < M^{\eta}_{ij}$, $m^{\xi}_{kl} < b$

Because of constraint α, $M^{\xi}_{ij} = a$ and because of constraint β, $m^{\eta}_{kl} = b$. $M^{\xi \Cup \eta}_{ij} = \max(M^{\xi}_{ij}, M^{\eta}_{ij}) = M^{\eta}_{ij}$ and $m^{\xi \Cup \eta}_{kl} = \min(m^{\xi}_{kl}, m^{\eta}_{kl}) = m^{\xi}_{kl}$. $I^{\xi \Cup \eta}_{ij} \setminus I^{\xi}_{ij} = \{u \in \mathbb{R}_{>0} \mid M^{\xi}_{ij} < u \leq M^{\eta}_{ij}\}$ is non-empty. So it is possible to pick a value $t_{ij} = M^{\xi}_{ij} + \delta$ ($\delta \in \mathbb{R}_{>0}$) in the interval $I^{\xi \Cup \eta}_{ij} \setminus I^{\xi}_{ij}$. $I^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{R}_{\geq 0} \mid m^{\xi}_{kl} \leq w < m^{\eta}_{kl}\}$ is non-empty.

(d) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \leq b$, $a < M^{\eta}_{ij}$, $b < M^{\xi}_{kl}$

Because of constraint α, $M^{\xi}_{ij} = a$ and because of constraint β, $M^{\eta}_{kl} = b$. $M^{\xi \Cup \eta}_{ij} = \max(M^{\xi}_{ij}, M^{\eta}_{ij}) = M^{\eta}_{ij}$ and $M^{\xi \Cup \eta}_{kl} = \max(M^{\xi}_{kl}, M^{\eta}_{kl}) = M^{\xi}_{kl}$. $I^{\xi \Cup \eta}_{ij} \setminus I^{\xi}_{ij} = \{u \in \mathbb{R}_{>0} \mid M^{\xi}_{ij} < u \leq M^{\eta}_{ij}\}$ is non-empty. So it is possible to pick a value $t_{ij} = M^{\xi}_{ij} + \delta$ ($\delta \in \mathbb{R}_{>0}$) in the interval $I^{\xi \Cup \eta}_{ij} \setminus I^{\xi}_{ij}$. $I^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl} = \{w \in \mathbb{R}_{\geq 0} \mid M^{\eta}_{kl} < w \leq M^{\xi}_{kl}\}$ is non-empty.

In all the four cases, after fixing the value of t_{ij} (i.e., setting $\boldsymbol{m}_{ij} = \boldsymbol{M}_{ij} = t_{ij}$ in interval $I^{\xi \Cup \eta}_{ij}$ of $\mathcal{D}^{\xi \Cup \eta}_{s}$ and stabilizing the table), $m^{\xi \Cup \eta}_{kl}$ and $M^{\xi \Cup \eta}_{kl}$ do not change, and therefore, $I^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl}$ is still non-empty. So it is always possible to pick a value for t_{kl} from $\boldsymbol{I}^{\xi \Cup \eta}_{kl} \setminus I^{\eta}_{kl}$.

Obviously, $t_{ij} \in \boldsymbol{I}^{\xi \Cup \eta}_{ij}$ and $t_{kl} \in \boldsymbol{I}^{\xi \Cup \eta}_{il}$. Moreover, $0 \leq i < j \leq k < l < n$, so, by Observation 3, it is possible to form a sequence $t_i \leq t_j \leq t_k \leq t_l$ that is compatible with $\boldsymbol{D}^{\xi \Cup \eta}_{s}$. By Theorem 1, S can be extended to a behaviour, say $\mathcal{B}^z$, in $[\![\xi \Cup \eta]\!]$. Clearly, $t^{\mathcal{B}^z}_{ij} \notin I^{\xi}_{ij}$ and $t^{\mathcal{B}^z}_{kl} \notin I^{\eta}_{kl}$. Therefore, $\mathcal{B}^z$ zigzags through ij and kl.

□