Operations on Timed Scenarios

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Abstract. We introduce three operations for timed scenarios: subsumption, intersection and union. They have constructive definitions based on the syntactic properties of the constituent scenarios. We show that they have the desired semantic properties. These results expand the theory of timed scenarios, but their development was inspired by practical considerations: they are directly relevant to the problem of synthesizing a timed automaton with a minimal number of clocks from a set of scenarios.

1 Introduction

Using scenarios for specification and implementation of complex systems (including real time systems), and synthesizing formal models of systems from scenarios have been active areas of research for several decades [1–9].

In our earlier work [10] we developed, from first principles, a formal, yet simple notation for timed scenarios. Intuitively, a scenario is a sequence of events along with a set of constraints between the times of these events. We defined the semantics of a timed scenario as the set of all *behaviours* (a.k.a timed words [11]) that are allowed by the scenario.

We want to use such scenarios to automatically synthesize formal models in the form of timed automata [11] with a *minimal number of clocks.* This is important, since the number of clocks crucially affects the cost of verification of timed automata [12].

As part of our earlier work [10] we obtained a canonical representation (a "stable distance table") for the entire class of scenarios that are equivalent to a given one. We used stable distance tables as a linchpin of various algorithms for determining the consistency and equivalency of scenarios, as well as for optimizing scenarios [13, 14].

In the current paper we use our stable distance tables once more to develop the notions of intersection, union and subsumption for timed scenarios. We introduce appropriate operations with well-defined semantics for computing the intersection and union of two consistent scenarios, as well as for determining whether a scenario is subsumed by another one.

Intuitively, given two consistent scenarios, their intersection is a scenario that expresses all those behaviours that are allowed by both, while their union is a scenario that captures all the behaviours that are allowed by either of them or both. A scenario is subsumed by another one if its set of behaviours is subsumed by that of the other one.

Having defined these operations, we develop some interesting relationships between them. In particular, we study the conditions under which the union of the behaviours allowed by two scenarios can be represented by a single scenario.

These operations are directly relevant to the problem of synthesizing timed automata from a set of scenarios, which is not addressed in the current paper.

2 Preliminaries

2.1 Timed automata

A *timed automaton* [11] is a tuple $\mathcal{A} = \langle \Sigma, Q, q_0, Q_f, C, T \rangle$, where Σ is a finite alphabet, Q is the (*finite*) set of locations, $q_0 \in Q$ is the initial location, $Q_f \subseteq Q$ is the set of final locations, C is a finite set of *clock* variables (clocks for short), and $T \subseteq Q \times Q \times \Sigma \times 2^C \times 2^{\Phi(C)}$ is the set of transitions. In each transition $(q, q', e, \lambda, \phi)$, λ is the set of clocks to be reset with the transition and $\phi \subset \Phi(C)$ is a set of clock constraints over C of the form $c \sim a$ (where $\sim \in \{\leq, <, \geq, >, =\}$, $c \in C$ and a is a constant in the set of rational numbers, $\mathbb{Q}$). A *clock valuation* ν for C is a mapping from C to $\mathbb{R}^{\geq 0}$. Clock valuation ν *satisfies* a set of clock constraints ϕ over C iff every clock constraint in ϕ evaluates to true after each clock c is replaced with $\nu(c)$. For $\tau \in \mathbb{R}, \nu + \tau$ denotes the clock valuation which maps every clock c to the value $\nu(c) + \tau$. For $Y \subseteq C$, $[Y \mapsto \tau]\nu$ is the valuation which assigns τ to each $c \in Y$ and agrees with ν over the rest of the clocks.

A *timed word* over an alphabet Σ is a pair (σ, τ) where $\sigma = \sigma_1\sigma_2...$ is a finite [15, 16] or infinite [11] word over Σ and $\tau = \tau_1\tau_2...$ is a finite or infinite sequence of (time) values such that (i) $\tau_i \in \mathbb{R}^{\geq 0}$, (ii) $\tau_i \leq \tau_{i+1}$ for all $i \geq 1$, and (iii) if the word is infinite, then for every $t \in \mathbb{R}^{\geq 0}$ there is some $i \geq 1$ such that $\tau_i > t$.

A run ρ of $\mathcal{A}$ over a timed word (σ, τ) is a finite or infinite sequence of the form $\langle q_0, \nu_0 \rangle \xrightarrow[\tau_1]{\sigma_1} \langle q_1, \nu_1 \rangle \xrightarrow[\tau_2]{\sigma_2} \langle q_2, \nu_2 \rangle \xrightarrow[\tau_3]{\sigma_3} \ldots$, where for all $i \geq 0$, $q_i \in Q$ and ν_i is a clock valuation such that (i) $\nu_0(c) = 0$ for all clocks $c \in C$ and (ii) for every $i > 1$ there is a transition in T of the form $(q_{i-1}, q_i, \sigma_i, \lambda_i, \phi_i)$, such that $(\nu_{i-1} + \tau_i - \tau_{i-1})$ satisfies ϕ_i, and ν_i equals $[\lambda_i \mapsto 0](\nu_{i-1} + \tau_i - \tau_{i-1})$.

A run over a finite timed word is *accepting* if it ends in a final location [16]. The *language* of $\mathcal{A}$, $L(\mathcal{A})$, is the set $\{(\sigma, \tau) \mid \mathcal{A}$ has an accepting run[1] over $(\sigma, \tau)\}$.

2.2 Timed scenarios

(This subsection briefly recounts our earlier work [10, 13].)

Let Σ be a finite set of symbols called *events*. A *behaviour* over Σ is a sequence $(e_0, t_0)(e_1, t_1)(e_2, t_2)\ldots$, such that $e_i \in \Sigma$, $t_i \in \mathbb{R}^{\geq 0}$ and $t_{i-1} \leq t_i$ for $i \in \{1, 2 \ldots\}$. For a finite behaviour $\mathcal{B} = (e_0, t_0)(e_1, t_1)\ldots(e_{n-1}, t_{n-1})$ of length n, and for any $0 \leq i < j < n$, the *distance*, in time units, of event j from event i in $\mathcal{B}$ is denoted by $t^{\mathcal{B}}_{ij}$. That is, $t^{\mathcal{B}}_{ij} = t_j - t_i$.

[1] In this work we only consider finite runs.

γ	η
$0 : a$;	$0 : a$;
$1 : b\ \{\tau_{0,1} \le 1\}$;	$1 : b\ \{\tau_{0,1} \le 1\}$;
$2 : c$;	$2 : c\ \{\tau_{0,2} \ge 3\}$;
$3 : d\ \{\tau_{0,3} = 5, \tau_{2,3} \le 2\}$.	$3 : d\ \{\tau_{0,3} = 5\}$.

Fig. 1: Two equivalent scenarios

	1	2	3
0	(0, 1)	(0, ∞)	(5,5)
1		(0, ∞)	(0, ∞)
2			(0, 2)

	1	2	3
0	(0, 1)	(3, 5)	(5, 5)
1		(2, 5)	(4, 5)
2			(0, 2)

Fig. 2: γ's initial table and its stable form

A *timed scenario* (*scenario* for short) of length $n \in \mathbb{N}$ over Σ is a pair $(\mathcal{E}, \mathcal{C})$, where $\mathcal{E} = e_o e_1 ... e_{n-1}$ is a sequence of events, and $\mathcal{C} \subset \Phi(n)$ is a finite set of constraints. Each constraint in $\Phi(n)$ is of the form $b \sim a$, where b is the symbol $\tau_{i,j}$ (for some integers $0 \le i < j < n$), $\sim \in \{\le, \ge\}$ [2] and a is a constant in the set of rational numbers, $\mathbb{Q}$. The intended interpretation is that $\tau_{i,j}$ is the time distance between the i-th and the j-th events in the behaviours described by a timed scenario.

A scenario will be written as a sequence of events, separated by semicolons and terminated by a period. If the scenario contains a constraint such as $\tau_{i,j} \le a$, then event j in the sequence will be accompanied by the constraint. We refer to this as the external representation of the scenario. Scenario γ in Fig. 1 is the external representation of scenario $(abcd, \{\tau_{0,1} \le 1, \tau_{0,3} = 5, \tau_{2,3} \le 2\})$.

A behaviour $\mathcal{B} = (e_0, t_0)(e_1, t_1) \ldots (e_{n-1}, t_{n-1})$ over Σ is *allowed* by scenario $\xi = (\mathcal{E}, \mathcal{C})$ iff $\mathcal{E} = e_0 \ldots e_{n-1}$ and every $\tau_{i,j} \sim a$ in $\mathcal{C}$ evaluates to true after $\tau_{i,j}$ is replaced by $t^{\mathcal{B}}_{ij}$. If $\mathcal{B}$ is allowed by ξ, then we say $\mathcal{B}$ satisfies all constraints of ξ.

The constraints $\tau_{i,j} \ge 0$ and $\tau_{i,j} \le \infty$, which always evaluate to true after we replace them with some $t^{\mathcal{B}}_{ij}$, will be called *default constraints*.

The *semantics* of scenario ξ, denoted by $[\![\xi]\!]$, is the set of behaviours that are allowed by ξ. For scenario γ in Fig. 1 $[\![\gamma]\!] = \{(a, t_0)(b, t_1)(c, t_2)(d, t_3) \mid t_3 \ge t_2 \ge t_1 \ge t_0 \wedge t_1 - t_0 \le 1 \wedge t_3 - t_0 = 5 \wedge t_3 - t_2 \le 2\}$.

A scenario ξ is *consistent* iff $[\![\xi]\!] \ne \emptyset$. It is *inconsistent* iff $[\![\xi]\!] = \emptyset$.

Two scenarios $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ are *equivalent* iff $[\![\xi]\!] = [\![\eta]\!]$. For example, γ and η of Fig. 1 are equivalent.

For a consistent scenario ξ of length n, and for $0 \le i < j < n$, $m^{\xi}_{ij} = min\{t^{\mathcal{B}}_{ij} \mid \mathcal{B} \in [\![\xi]\!]\}$ and $M^{\xi}_{ij} = max\{t^{\mathcal{B}}_{ij} \mid \mathcal{B} \in [\![\xi]\!]\}$. The absence of an upper bound for some i and j will be denoted by $M^{\xi}_{ij} = \infty$. We will write just m_{ij} and M_{ij} when ξ is understood. For any behaviour in $[\![\xi]\!]$, $0 \le m_{ij} \le t_{ij} \le M_{ij} \le \infty$.

For a consistent scenario ξ of length n, and for any $0 \le i < j < k < n$ the following inequations hold:

$$m_{ij} + m_{jk} \le m_{ik} \le \left\{ \begin{matrix} m_{ij} + M_{jk} \\ M_{ij} + m_{jk} \end{matrix} \right\} \le M_{ik} \le M_{ij} + M_{jk} \tag{1}$$

[2] To keep the presentation compact, sharp inequalities are not allowed [10]. Equality is expressed in terms of $\le$ and $\ge$.

Let $\xi = (\mathcal{E}, \mathcal{C})$ be a scenario of length n, such that, for any $0 \le i < j < n$, $\mathcal{C}$ contains at most one constraint of the form $\tau_{i,j} \ge c$ and at most one of the form $\tau_{i,j} \le c$. A *distance table* for ξ is a representation of $\mathcal{C}$ in the form of a triangular matrix $\mathcal{D}^\xi$. For $0 \le i < j < n$, $\mathcal{D}^\xi[i,j] = (l^\xi_{ij}, h^\xi_{ij})$, where l^ξ_{ij} and h^ξ_{ij} are rational numbers. If $\tau_{i,j} \ge c \in \mathcal{C}$ then $l_{ij} = c$, otherwise $l_{ij} = 0$; if $\tau_{i,j} \le c \in \mathcal{C}$ then $h_{ij} = c$, otherwise $h_{ij} = \infty$. We will write just l_{ij} and h_{ij} when ξ is understood. The distance table corresponding to γ of Fig. 1 is shown on the left of Fig. 2.

A distance table of size n is *valid* iff $l_{ij} \le h_{ij}$, for all $0 \le i < j < n$. A table that is not valid is *invalid*. If $\mathcal{D}^\xi$ is invalid, then ξ is obviously inconsistent.

A valid distance table of size n is *stable* iff, for all $0 \le i < j < k < n$, the inequations in (1) hold when m_{ij}, m_{jk}, m_{ik} are replaced by l_{ij}, l_{jk}, l_{ik} and M_{ij}, M_{jk}, M_{ik} are replaced by h_{ij}, h_{jk}, h_{ik}. If $\mathcal{D}^\xi$ is stable then ξ is consistent.

To *stabilise* $\mathcal{D}^\xi$ the following six rules are repeatedly applied until the table becomes either invalid or stable [10]. The stabilized table is denoted by $\mathcal{D}^\xi_s$.

$$
\begin{array}{ll}
l_{ij} + l_{jk} > l_{ik} \longrightarrow l_{ik} := l_{ij} + l_{jk} & l_{ik} > l_{ij} + h_{jk} \longrightarrow l_{ij} := l_{ik} - h_{jk} \\
l_{ik} > h_{ij} + l_{jk} \longrightarrow l_{jk} := l_{ik} - h_{ij} & l_{ij} + h_{jk} > h_{ik} \longrightarrow h_{jk} := h_{ik} - l_{ij} \\
h_{ij} + l_{jk} > h_{ik} \longrightarrow h_{ij} := h_{ik} - l_{jk} & h_{ik} > h_{ij} + h_{jk} \longrightarrow h_{ik} := h_{ij} + h_{jk}
\end{array}
$$

At least one of these rules is applicable if and only if some inequation in (1) does not hold. The purpose of each rule is to tighten a constraint just enough to establish a particular inequation. A stable distance table[3] has two properties. First, as a result of applying the rules above, the table includes all the constraints that are "implied" by the initial set of constraints. Second, all the constraints represented by the table are as *tight* as possible:

Observation 1 *Let ξ be a scenario of length n. If $\mathcal{D}^\xi$ is stable, then for every $0 \le i < j < n$, $l_{ij} = m_{ij}$ and $h_{ij} = M_{ij}$.*

In other words, the pair of values (m_{ij}, M_{ij}) in entry $[i,j]$ of a stable distance table, i.e., $\mathcal{D}^\xi_s[i,j] = (m^\xi_{ij}, M^\xi_{ij})$, specifies the interval of all the possible values of t_{ij} that can appear in the behaviours allowed by the corresponding scenario.

Two scenarios ξ and η are equivalent iff $\mathcal{D}^\xi_s = \mathcal{D}^\eta_s$. The table on the right-hand side of Fig. 2 is the stable distance table obtained from the constraints of either γ or η (which shows that they are equivalent: $[\![\gamma]\!] = [\![\eta]\!]$).

Given a scenario ξ with its stable distance table $\mathcal{D}^\xi_s$, we use $\mathcal{C}(\mathcal{D}^\xi_s)$ to denote the set of constraints represented by $\mathcal{D}^\xi_s$.

Definition 1. *Let ξ be a scenario of length n, $\mathcal{D}^\xi_s$ be its stable table, $c \in \mathcal{C}(\mathcal{D}^\xi_s)$ be a non-default constraint, $S \subset \mathcal{C}(\mathcal{D}^\xi_s)$, and $0 \le i < j < k < n$. Constraint c is* directly supported *by S, denoted by $S \leadsto c$, iff c and S satisfy one of the following six conditions:*

[3] The stable distance tables, though derived in a very different fashion, turned out to be essentially equivalent to Dill's Difference Bounds Matrices (DBMs) [17]. However, when applied to the particular case of scenarios, the constraint reduction technique for DBMs [18] is weaker than the original optimisation algorithm [13] and also the one developed later [14]. A detailed comparison can be found in the cited work [13].

1. $c = \tau_{i,k} \geq u$, $S = \{\tau_{i,j} \geq v, \tau_{j,k} \geq w\}$, *and* $u = v + w$.
2. $c = \tau_{i,j} \geq u$, $S = \{\tau_{i,k} \geq v, \tau_{j,k} \leq w\}$, *and* $u = v - w$.
3. $c = \tau_{j,k} \geq u$, $S = \{\tau_{i,k} \geq v, \tau_{i,j} \leq w\}$, *and* $u = v - w$.
4. $c = \tau_{j,k} \leq u$, $S = \{\tau_{i,k} \leq v, \tau_{i,j} \geq w\}$, *and* $u = v - w$.
5. $c = \tau_{i,j} \leq u$, $S = \{\tau_{i,k} \leq v, \tau_{j,k} \geq w\}$, *and* $u = v - w$.
6. $c = \tau_{i,k} \leq u$, $S = \{\tau_{i,j} \leq v, \tau_{j,k} \leq w\}$, *and* $u = v + w$.

Each of the cases in Definition 1 corresponds to one of the six rules of stabilization mentioned before. For example, if $m_{13} = 3$, $M_{36} = 4$ and $m_{16} = 0$ (i.e., the corresponding constraint is missing), then the first rule will force m_{16} to be 7: the constraint $m_{16} = 7$ is directly supported (i.e., "implied") by the other two.

Definition 2. *(quasi-transitivity) Let $\mathcal{D}_s^\xi$ be a stable table. $\rightsquigarrow^+ \subset 2^{\mathcal{C}(\mathcal{D}_s^\xi)} \times \mathcal{C}(\mathcal{D}_s^\xi)$ is the smallest relation that satisfies the following two conditions:*
1. *If $S \rightsquigarrow c$ then $S \rightsquigarrow^+ c$;*
2. *If $S \rightsquigarrow^+ c$ and there is a constraint $d \in S$ such that, for some $S' \in \mathcal{C}(\mathcal{D}_s^\xi)$, $S' \rightsquigarrow^+ d$ and $c \notin S'$, then $(S \setminus \{d\}) \cup S' \rightsquigarrow^+ c$.*

Constraint c is supported *by S when $S \rightsquigarrow^+ c$. S is then called a* support *of c.*

Intuitively, if a constraint d has a support, then d can be removed from the scenario. The removed d can be a member of the supports of other constraints, e.g., d can appear in a set S that supports c. As long as d has a support S' that does not include c, S can be updated by replacing d with S'. The relation $\rightsquigarrow^+$ captures all the possible supports for the constraints in $\mathcal{C}(\mathcal{D}_s^\xi)$.

Observation 2 *If $S \rightsquigarrow^+ c$, then $c \notin S$.*

Observation 3 *Let $S \rightsquigarrow^+ c$. If behaviour $\mathcal{B}$ satisfies all the constraints in S, then it also satisfies c.*

2.3 Timed scenarios and timed automata

If $\xi = (\mathcal{E}, \mathcal{C})$ is a scenario of length n, and $\mathcal{C}$ contains a constraint $\tau_{i,j} \sim a$ for some $0 \leq i < j < n$, and some $a \in \mathbb{Q}$, then the index i is an *anchor*. For an anchor i, if $0 < j < n$ is the largest number such that $\tau_{i,j} \sim a$ is a constraint in $\mathcal{C}$, then $[i, j)$ is the *range* of anchor i. If i_1 and i_2 are two anchors with ranges $[i_1, j_1)$ and $[i_2, j_2)$ in ξ, then the two ranges *overlap* iff $i_1 < i_2 < j_1$ or $i_2 < i_1 < j_2$.

For example, in scenario γ of Fig. 1, the range of anchor 0 is $[0, 3)$ and the range of anchor 2 is $[2, 3)$: these are overlapping.

$Anch_\xi$ is used to denote the set of anchors of ξ. If X is a set of clock variables, then the relation $alloc_\xi \subset Anch_\xi \times X$ is a clock allocation for ξ. $alloc_\xi$ is *complete* iff for every anchor $i \in Anch_\xi$ there is a clock $x \in X$ such that $(i, x) \in alloc_\xi$. $alloc_\xi$ is *incorrect* iff there exist two different anchors i and j in $Anch_\xi$ whose ranges overlap, such that $(i, x) \in alloc_\xi$ and $(j, x) \in alloc_\xi$ for some $x \in X$. $alloc_\xi$ is *correct* iff it is not incorrect. A correct and complete clock allocation is *optimal* if there is no other correct and complete allocation that uses fewer clocks. $\{(0, x), (2, y)\}$ is an optimal clock allocation for scenario γ of Fig. 1.

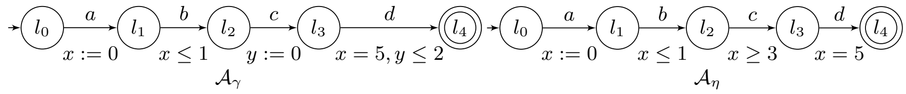

Fig. 3: Equivalent timed automata corresponding to the scenarios of Fig. 1

A scenario ξ can be trivially converted to a simple timed automaton $\mathcal{A}_\xi$: the language of $\mathcal{A}_\xi$ is equivalent to the set of behaviours allowed by ξ: $L(\mathcal{A}_\xi) = [\![\xi]\!]$.

Given a scenario ξ, it is possible to transform it to an equivalent scenario η such that $\mathcal{A}_\eta$ has the *minimal* number of clocks in the entire class of timed automata that are obtained from all scenarios equivalent to ξ [14]. Then η is the *optimized* form of ξ.

For example, scenario η of Fig. 1 is the optimized form of γ in that figure. Their corresponding language-equivalent timed automata are shown in Fig. 3. Notice that $\mathcal{A}_\eta$ has only one clock, while $\mathcal{A}_\gamma$ requires two clocks.

3 Operations on timed scenarios and synthesis problem

In this section we briefly present the motivation behind developing the notions of *subsumption*, *intersection*, and *union* for timed scenarios.

Given a set Ξ of scenarios, our ultimate goal is to synthesize a timed automaton $\mathcal{A}_\Xi$ whose language would be $\bigcup_{\xi \in \Xi} [\![\xi]\!]$ [4]. We want $\mathcal{A}_\Xi$ to have the minimal number of clocks in the entire class of language-equivalent timed automata.

Let Ξ_o be the set of optimized scenarios obtained from the members of Ξ. Let $\Gamma \subseteq \Xi_o$ be a finite set of those members of Ξ_o that have identical sequence of events. The members of Γ may be involved in interesting relationships that may have a bearing on the best way to reach our goal.

Subsumption If Γ contains two scenarios ξ and η, such that $[\![\xi]\!] \subseteq [\![\eta]\!]$, then ξ can be removed from Γ. This cannot increase the number of clocks in $\mathcal{A}_\Xi$, but may decrease it, so such subsumed scenarios should always be removed.

For example, let ξ and γ of Fig. 4 be in Γ. Notice that $[\![\xi]\!] \subseteq [\![\gamma]\!]$, that is, $\xi \subseteq \gamma$ (see Definition 4 in Sec. 4). So scenario ξ must be discarded. This removal will result in a reduction in the number of clocks in the synthesized automaton: the two anchors in ξ have overlapping ranges, so $\mathcal{A}_\xi$ would require two clocks, but $\mathcal{A}_\gamma$ would only require one clock.

Union The set Γ may contain a subset S, such that $|S| > 1$, but all its members can be replaced by a single scenario. More precisely, there may exist a scenario η such that $[\![\eta]\!] = \bigcup_{\xi \in S} [\![\xi]\!]$. We then say that the members of S can be *combined.* Combining scenarios can, in general, result in a reduction in the number of clocks in the synthesized automaton. We discuss this for a very simple case.

[4] The complete account of the synthesis problem is addressed in a forthcoming paper.

ξ	$0 : a$; $1 : b$; $2 : c\ \{\tau_{1,2} \leq 5\}$; $3 : d\ \{\tau_{0,3} \geq 7\}$.	η	$0 : a$; $1 : b$; $2 : c\ \{\tau_{1,2} \geq 5\}$; $3 : d\ \{\tau_{0,3} \geq 7\}$.	γ	$0 : a$; $1 : b$; $2 : c$; $3 : d\ \{\tau_{0,3} \geq 7\}$.

Fig. 4: Two scenarios ξ and η with complementary constraints, and their union

Let $\xi = (\mathcal{E}, \mathcal{C}^{\xi})$ and $\eta = (\mathcal{E}, \mathcal{C}^{\eta})$ in Γ be such that they can be combined into one scenario. Let the scenarios contain a pair of explicit constraints that are complementary, i.e., for some i, j and a we have $\tau_{i,j} \leq a \in \mathcal{C}^{\xi}$ and $\tau_{i,j} \geq a \in \mathcal{C}^{\eta}$. Then in the combined scenario (see Definition 6 in Sec. 4) the only constraints on $\tau_{i,j}$ will be implicit default constraints.

As an example consider scenarios ξ and η of Fig. 4. These two scenarios can be combined: $[\![\xi]\!] \cup [\![\eta]\!]$ is captured by scenario γ in that figure. Notice that ξ and η each have two anchors 0 and 1 whose ranges overlap. Therefore $\mathcal{A}_{\xi}$ and $\mathcal{A}_{\eta}$ each would require two clocks. So the number of clocks in the synthesized automaton (from ξ and η) cannot be smaller than 2. However, a language-equivalent automaton can be synthesized from γ that would require only one clock: by combining ξ and η we can get rid of the pair of complementary constraints, i.e., $\tau_{1,2} \leq 5$ and $\tau_{1,2} \geq 5$ (see γ on the right). Anchor 1 disappears entirely, so $\mathcal{A}_{\gamma}$ would require only one clock.

In general, there are two situations where combining scenarios results in a reduction in the number of clocks in a synthesized automaton. One is when an anchor i disapears entirely, and another one is when the number of ranges that overlap with range of i decreases.

So we should try to combine scenarios with identical sequences of events, before we begin the synthesis process.

Intersection Γ might contain two scenarios ξ and η, such that $[\![\xi]\!] \cap [\![\eta]\!] \neq \emptyset$. In that case, the intersection can be expressed by a timed scenario.

Computing the intersection of a set of scenarios is not directly relevant to our task, but will be useful in computing their union (see Sec. 4 for details).

Next, we formally define these three operations.

4 Subsumption, intersection and union

For a given scenario $\xi = (\mathcal{E}, \mathcal{C})$ the members of $\mathcal{C}$ will be referred to as *explicit constraints*. We know that in the stable distance table, $\mathcal{D}_s^{\xi}$, there are also *implicit* constraints: default constraints and constraints that are implied by $\mathcal{C}$.

Definition 3. *Let ξ be a scenario of length n and let $\mathcal{D}_s^{\xi}$ be its stable distance table. Then, for any $0 \leq i < j < n$, the interval I_{ij}^{ξ} is $\{a \in \mathbb{Q} \mid m_{ij}^{\xi} \leq a \leq M_{ij}^{\xi}\}$ when $\mathcal{D}_s^{\xi}[i, j] = (m_{ij}^{\xi}, M_{ij}^{\xi})$.*

Intuitively, I^{ξ}_{ij} corresponds to a pair of constraints: the time distance between events i and j in every behaviour in $[\![\xi]\!]$ must be at least m^{ξ}_{ij} and at most M^{ξ}_{ij}.

Definition 4. *Let ξ and η be two consistent scenarios with the same sequence of events. We say ξ is* subsumed *by η, denoted by $\xi \subseteq \eta$, when $[\![\xi]\!] \subseteq [\![\eta]\!]$.*

Observation 4 *Let ξ and η be two consistent scenarios of length n with the same sequence of events. Then $\xi \subseteq \eta$ iff $\forall_{0 \leq i<j<n} I^{\xi}_{ij} \subseteq I^{\eta}_{ij}$.*

Thanks to Observation 4, given two scenarios it can be easily checked, in quadratic time, whether one is subsumed by the other one.

Definition 5. *Let ξ and η be two consistent scenarios of length n with the same sequence of events, $\mathcal{E}$, such that $\forall_{0 \leq i<j<n}\ I^{\xi}_{ij} \cap I^{\eta}_{ij} \neq \emptyset$. The* intersection *of ξ and η, denoted by $\xi \cap \eta$, is a scenario whose sequence of events is $\mathcal{E}$ and $\mathcal{D}^{\xi\cap\eta}[i,j] = (\max(m^{\xi}_{ij}, m^{\eta}_{ij}), \min(M^{\xi}_{ij}, M^{\eta}_{ij}))$.*

If $\exists_{0 \leq k<l<n} I^{\xi}_{kl} \cap I^{\eta}_{kl} = \emptyset$, then $\xi \cap \eta$ is not defined.

Assume $\mathcal{D}^{\xi}_{s}[i,j] = (a_1, b_1)$, $\mathcal{D}^{\eta}_{s}[i,j] = (a_2, b_2)$, and $\xi \cap \eta$ is defined. Then, $\mathcal{D}^{\xi\cap\eta}[i,j] = (l^{\xi\cap\eta}_{ij}, h^{\xi\cap\eta}_{ij}) = (\max(a_1, a_2), \min(b_1, b_2))$. Since $I^{\xi}_{ij} \cap I^{\eta}_{ij} \neq \emptyset$, we have $\max(a_1, a_2) \leq \min(b_1, b_2)$. So the initial table is always valid. But it might not be stable, so we must stabilise it[5] to check whether $\xi \cap \eta$ is consistent. In the resulting stable table $m^{\xi\cap\eta}_{ij} \geq l^{\xi\cap\eta}_{ij}$ and $M^{\xi\cap\eta}_{ij} \leq h^{\xi}_{ij}$ for every $0 \leq i < j < n$, because stabilisation only tightens the constraints.

Figures 5 and 6 show two scenarios, ξ and η, along with their stable distance tables. Observe that $I^{\xi}_{ij} \cap I^{\eta}_{ij} \neq \emptyset$, for $0 \leq i < j \leq 2$. Fig. 7 shows the initial table corresponding to their intersection and its stabilized form.

Lemma 1. *Let ξ and η be two consistent scenarios such that $[\![\xi]\!] \cap [\![\eta]\!] \neq \emptyset$. Then the scenario $\xi \cap \eta$ is defined and consistent. Moreover, $[\![\xi]\!] \cap [\![\eta]\!] \subseteq [\![\xi \cap \eta]\!]$.*

Proof. Let $\mathcal{B} \in [\![\xi]\!] \cap [\![\eta]\!]$. Then

$\forall_{0 \leq i<j<n}\ t^{\mathcal{B}}_{ij} \leq M^{\xi}_{ij} \wedge t^{\mathcal{B}}_{ij} \leq M^{\eta}_{ij}$. So $t^{\mathcal{B}}_{ij} \leq \min(M^{\xi}_{ij}, M^{\eta}_{ij})$.

$\forall_{0 \leq i<j<n}\ t^{\mathcal{B}}_{ij} \geq m^{\xi}_{ij} \wedge t^{\mathcal{B}}_{ij} \geq m^{\eta}_{ij}$. So $t^{\mathcal{B}}_{ij} \geq \max(m^{\xi}_{ij}, m^{\eta}_{ij})$.

Therefore $\forall_{0 \leq i<j<n}\ \max(m^{\xi}_{ij}, m^{\eta}_{ij}) \leq \min(M^{\xi}_{ij}, M^{\eta}_{ij})$. So $\xi \cap \eta$ is defined.

Behaviour $\mathcal{B}$ satisfies all the constraints represented by the initial table of $\xi \cap \eta$, i.e., $\mathcal{D}^{\xi\cap\eta}$ (see Definition 5). Stabilization does not remove any behaviour, so $\mathcal{B}$ satisfies also all the constraints represented by $\mathcal{D}^{\xi\cap\eta}_{s}$. That is, $\mathcal{B} \in [\![\xi \cap \eta]\!]$. Therefore $[\![\xi \cap \eta]\!] \neq \emptyset$, and hence $\xi \cap \eta$ is consistent. Moreover, $[\![\xi]\!] \cap [\![\eta]\!] \subseteq [\![\xi \cap \eta]\!]$.

Theorem 1. *Let ξ and η be two consistent scenarios such that $\xi \cap \eta$ is defined and consistent. Then $[\![\xi \cap \eta]\!] = [\![\xi]\!] \cap [\![\eta]\!]$.*

Proof. Assume the length of ξ and η is n, and let $\mathcal{B} \in [\![\xi \cap \eta]\!]$. Then

$\forall_{0 \leq i<j<n}\ t^{\mathcal{B}}_{ij} \geq m^{\xi\cap\eta}_{ij} \geq l^{\xi\cap\eta}_{ij} = \max(m^{\xi}_{ij}, m^{\eta}_{ij})$. So $t^{\mathcal{B}}_{ij} \geq m^{\xi}_{ij}$ and $t^{\mathcal{B}}_{ij} \geq m^{\eta}_{ij}$.

[5] The cost is $O(n^3)$, where n is the length of the scenario [10].

$0 : a$;
$1 : b\ \{\tau_{0,1} \leq 2\}$;
$2 : c\ \{\tau_{0,2} \geq 5\}$.

	1	2
0	$(0,2)$	$(5,\infty)$
1		$(3,\infty)$

Fig. 5: ξ and its stable distance table

$0 : a$;
$1 : b$;
$2 : c\ \{\tau_{1,2} =< 3\}$.

	1	2
0	$(0,\infty)$	$(0,\infty)$
1		$(0,3)$

Fig. 6: η and its stable distance table

	1	2
0	$(0,2)$	$(5,\infty)$
1		$(3,3)$

	1	2
0	$(2,2)$	$(5,5)$
1		$(3,3)$

Fig. 7: The initial table and its stabilized form for $\xi \cap \eta$

$0 : a$;
$1 : b\ \{\tau_{0,1} = 2\}$;
$2 : c\ \{\tau_{1,2} = 3\}$.

Fig. 8: $\xi \cap \eta$

$\forall_{0 \leq i<j<n}\ t^{\mathcal{B}}_{ij} \leq M^{\xi\cap\eta}_{ij} \leq h^{\xi\cap\eta}_{ij} = \min(M^{\xi}_{ij}, M^{\eta}_{ij})$. So $t^{\mathcal{B}}_{ij} \leq M^{\xi}_{ij}$ and $t^{\mathcal{B}}_{ij} \leq M^{\eta}_{ij}$. Therefore $\mathcal{B} \in [\![\xi]\!]$ and $\mathcal{B} \in [\![\eta]\!]$, that is, $\mathcal{B} \in [\![\xi]\!] \cap [\![\eta]\!]$. So $[\![\xi \cap \eta]\!] \subseteq [\![\xi]\!] \cap [\![\eta]\!]$.

Since $[\![\xi \cap \eta]\!] \neq \emptyset$, it follows that $[\![\xi]\!] \cap [\![\eta]\!] \neq \emptyset$ and, by Lemma 1, $[\![\xi]\!] \cap [\![\eta]\!] \subseteq [\![\xi \cap \eta]\!]$. So $[\![\xi \cap \eta]\!] = [\![\xi]\!] \cap [\![\eta]\!]$.

Consider ξ and η of figures 5 and 6 once more. Clearly, in the set of behaviours that are allowed by both ξ and η the time distance between a and b, and between b and c must be exactly 2 and 3, respictively. This set is captured by scenario $\xi \cap \eta$, shown in Fig. 8, which is obtained from the stabilized table of Fig. 7.

Definition 6. *Let ξ and η be two consistent scenarios of length n with the same sequence of events, $\mathcal{E}$, such that $\forall_{0 \leq i<j<n}\ I^{\xi}_{ij} \cap I^{\eta}_{ij} \neq \emptyset$. The* combination *of ξ and η, denoted by $\xi \Cup \eta$, is a scenario whose sequence of events is $\mathcal{E}$ and $\mathcal{D}^{\xi\Cup\eta}[i,j] = (\min(m^{\xi}_{ij}, m^{\eta}_{ij}), \max(M^{\xi}_{ij}, M^{\eta}_{ij}))$.*

If $\exists_{0 \leq i<j<n}\ I^{\xi}_{ij} \cap I^{\eta}_{ij} = \emptyset$, then $I^{\xi}_{ij} \cup I^{\eta}_{ij}$ does not constitute a single interval. In that case ξ and η cannot be combined, and so $\xi \Cup \eta$ is not defined.

Assume $\mathcal{D}^{\xi}_{s}[i,j] = (a_1, b_1)$, $\mathcal{D}^{\eta}_{s}[i,j] = (a_2, b_2)$, and $\xi \Cup \eta$ is defined. Then $\mathcal{D}^{\xi\Cup\eta}[i,j] = (l^{\xi\Cup\eta}_{ij}, h^{\xi\Cup\eta}_{ij}) = (\min(a_1, a_2), \max(b_1, b_2))$. Clearly, $\mathcal{D}^{\xi\Cup\eta}$ is stable, since all the inequations in 1 are satisfied by its content (see Observation 1):

Observation 5 *If $\xi \Cup \eta$ is defined, then $\mathcal{D}^{\xi\Cup\eta} = \mathcal{D}^{\xi\Cup\eta}_{s}$. So $\xi \Cup \eta$ is consistent.*

Lemma 2. *Let ξ and η be two consistent scenarios such that $\xi \Cup \eta$ is defined. Then $[\![\xi]\!] \cup [\![\eta]\!] \subseteq [\![\xi \Cup \eta]\!]$.*

Proof. ξ is consistent, so $[\![\xi]\!] \neq \emptyset$. Let $\mathcal{B} \in [\![\xi]\!]$. Then

$\forall_{0 \leq i<j<n}\ t^{\mathcal{B}}_{ij} \geq m^{\xi}_{ij} \geq \min(m^{\xi}_{ij}, m^{\eta}_{ij}) = l^{\xi\Cup\eta}_{ij}$, and

$\forall_{0 \leq i<j<n}\ t^{\mathcal{B}}_{ij} \leq M^{\xi}_{ij} \leq \max(M^{\xi}_{ij}, M^{\eta}_{ij}) = h^{\xi\Cup\eta}_{ij}$.

Then $t^{\mathcal{B}}_{ij} \in I^{\xi\Cup\eta}_{ij}$. Therefore $\mathcal{B}$ satisfies all the constraints represented by $\mathcal{D}^{\xi\Cup\eta}$, which is stable by Observation 5. So $\mathcal{B} \in [\![\xi \Cup \eta]\!]$. Therefore $[\![\xi]\!] \cup [\![\eta]\!] \subseteq [\![\xi \Cup \eta]\!]$.

A similar argument can be made if $\mathcal{B} \in [\![\eta]\!]$.

ξ

$0 : a \;;$
$1 : b \; \{\tau_{0,1} \leq 4\} \;;$
$2 : c \; .$

	1	2
0	(0, 4)	(0, ∞)
1		(0, ∞)

η

$0 : a \;;$
$1 : b \;;$
$2 : c \; \{\tau_{0,2} \geq 7\} \; .$

	1	2
0	(0, ∞)	(7, ∞)
1		(0, ∞)

Fig. 9: Two scenarios, ξ and η, and their stable tables

γ

$0 : a \;;$
$1 : b \;;$
$2 : c \; .$

	1	2
0	(0, ∞)	(0, ∞)
1		(0, ∞)

ζ

$0 : a \;;$
$1 : b \; \{\tau_{0,1} \geq 5\} \;;$
$2 : c \; \{\tau_{0,2} \leq 6\} \; .$

	1	2
0	(5, 6)	(5, 6)
1		(0, 1)

Fig. 10: The combination of ξ and η of Fig. 9, and scenario ζ with its stabilized table

Table $\mathcal{D}_s^{\xi \Cup \eta}$ allows all the behaviours in $[\![\xi]\!] \cup [\![\eta]\!]$. But there is a possibility that it may also allow some extra behaviours, namely those that satisfy all the constraints of the union, but do not satisfy some of the constraints in ξ and some of the constraints in η. We call such behaviours "zigzagging" behaviours.

It is not too difficult to find such zigzagging behaviours, if they exist. We discuss this for the simple case of an attempt to combine two scenarios ξ and η into a scenario γ.

We identify two constraints α and β such that α cannot be satisfied by behaviours in $[\![\xi]\!]$, but is satisfied by the members of $[\![\eta]\!]$, and β cannot be satisfied by behaviours in $[\![\eta]\!]$, but is satisfied by the members of $[\![\xi]\!]$. We then construct a scenario ζ with the same sequence of events as ξ and η, and set of constraints that includes only α and β. If ζ is consistent, we compute its intersection with γ. If the intersection is not empty, then γ is too permissive, since none of the behaviours allowed by ζ is allowed by ξ or η.

As an example consider two scenarios, ξ and η, of Fig. 9 along with their stabilized tables. Scenario γ of Fig. 10 shows their supposedly combined scenario along with its stabilized table. Scenario ζ of Fig. 10 represents a set of behaviours in which the time distance between events a and b are at least 5, and between events a and c are at most 6 units of time. There is no behaviour in the semantics of ζ that is allowed by either ξ or η of Fig. 9. Yet, its intersection with γ (the supposed union of ξ and η) is not empty. *This indicates that the union of the behaviours allowed by ξ and η cannot be represented by a single scenario.*

As another example consider ξ and η of Fig. 11 where the union of behaviours *can be* represented by a single scenario, namely their union (see the scenario on the right). That is, no "zigzagging" behaviour is allowed by $\xi \Cup \eta$: every behaviour in $[\![\xi \Cup \eta]\!]$ is in $[\![\xi]\!]$ or in $[\![\eta]\!]$ or in both.

The union of two scenarios ξ and η can be expressed by a single scenario if there is no "zigzagging" behaviours possible between ξ and η.

Next, we formalize "zigzagging" behaviours and define the union of scenarios.

ξ	η	$\xi \Cup \eta$
$0 : a$;	$0 : a$;	$0 : a$;
$1 : b$;	$1 : b$;	$1 : b$;
$2 : c\ \{\tau_{0,2} \geq 1, \tau_{0,2} \leq 6\}$;	$2 : c\ \{\tau_{0,2} \geq 4, \tau_{0,2} \leq 9\}$;	$2 : c\ \{\tau_{0,2} \geq 1, \tau_{0,2} \leq 9\}$;
$3 : d$;	$3 : d$;	$3 : d$;
$4 : e\ \{\tau_{1,4} \leq 8\}$.	$4 : e\ \{\tau_{1,4} \leq 8\}$.	$4 : e\ \{\tau_{1,4} \leq 8\}$.

Fig. 11: Two scenarios that can be combined

Definition 7. *Let ξ and η be two consistent scenarios of length n with the same sequence of events, $\mathcal{E}$, such that $\xi \Cup \eta$ is defined. We define $\mathcal{Z}(\xi, \eta)$ to be a (possibly empty) set of scenarios. Scenario $\zeta \in \mathcal{Z}(\xi, \eta)$ iff ζ is consistent and*
1. *The sequence of events of ζ is $\mathcal{E}$,*
2. *For every $0 \leq i < j < n$, $I_{ij}^{\zeta} \subseteq I_{ij}^{\xi \Cup \eta}$,*
3. *There exist $0 \leq i < j < n$ and $0 \leq k < l < n$ such that*
   (a) $i \neq k$ or $l \neq j$, and
   (b) $I_{ij}^{\xi} \cap I_{ij}^{\zeta} = \emptyset$, $I_{ij}^{\eta} \cap I_{ij}^{\zeta} \neq \emptyset$, and
   (c) $I_{kl}^{\xi} \cap I_{kl}^{\zeta} \neq \emptyset$, $I_{kl}^{\eta} \cap I_{kl}^{\zeta} = \emptyset$.

Lemma 3. *Let ξ and η be two consistent scenarios, such that $\xi \Cup \eta$ is defined. If $\zeta \in \mathcal{Z}(\xi, \eta)$, then $[\![\zeta]\!] \cap [\![\xi \Cup \eta]\!] \neq \emptyset$.*

Proof. By Definition 7, $\xi \Cup \eta$ is defined. So, by Observation 5, it is consistent. Therefore $[\![\xi \Cup \eta]\!] \neq \emptyset$. By Definition 7, ζ is consistent, so $[\![\zeta]\!] \neq \emptyset$.

By pt. 2 of Definition 7, $\forall_{0 \leq i < j < n}\ I_{ij}^{\zeta} \subseteq I_{ij}^{\xi \Cup \eta}$. So, by Observation 4, $[\![\zeta]\!] \subseteq [\![\xi \Cup \eta]\!]$. Therefore $[\![\xi \Cup \eta]\!] \cap [\![\zeta]\!] \neq \emptyset$.

Recall that every interval of the stable table, e.g., I_{ij}^{ξ} of scenario $\xi = (\mathcal{E}, \mathcal{C})$, corresponds to a pair of constraints. These constraints could be implicit (default and implied) constraints, or explicit constraints, i.e., members of $\mathcal{C}$.

Intuitively, if $\zeta \in \mathcal{Z}(\xi, \eta)$, then there must be at least two different constraints (explicit or implicit, non-default) one in ξ and one in η that are not satisfied by the members of $[\![\zeta]\!]$. Equivalently:

Observation 6 *Let $\zeta \in \mathcal{Z}(\xi, \eta)$. Then, for every $\mathcal{B} \in [\![\zeta]\!]$, there exist $0 \leq i < j < n$, and $0 \leq k < l < n$, where $i \neq k$ or $l \neq j$, such that $t_{ij}^{\mathcal{B}} \notin I_{ij}^{\xi}$, and $t_{kl}^{\mathcal{B}} \notin I_{kl}^{\eta}$.*

Lemma 4. *Let ξ and η be two consistent scenarios. If $\zeta \in \mathcal{Z}(\xi, \eta)$, then $[\![\zeta]\!] \cap [\![\xi]\!] = \emptyset$ and $[\![\zeta]\!] \cap [\![\eta]\!] = \emptyset$.*

Proof. ξ is consistent, so $[\![\xi]\!] \neq \emptyset$. By Definition 7, ζ is consistent, so $[\![\zeta]\!] \neq \emptyset$.

Assume $[\![\zeta]\!] \cap [\![\xi]\!] \neq \emptyset$. Then, by Lemma 1, $\zeta \cap \xi$ is defined and consistent. So, by Definition 5, $\forall_{0 \leq i < j < n}\ I_{ij}^{\zeta} \cap I_{ij}^{\xi} \neq \emptyset$. But this is contradictory to pt. 3 of Definition 7.

The proof for $[\![\zeta]\!] \cap [\![\eta]\!] = \emptyset$ is identical.

Observation 7 *Let ξ and η be two consistent scenarios. $\xi \subseteq \eta$ iff $\xi \Cup \eta = \eta$.*

Definition 8. *A scenario $\xi = (\mathcal{E}, \mathcal{C})$ is* trivial *if $\mathcal{C} = \emptyset$. ξ is* non-trivial *if it is not trivial.*

Intuitively, all the constraints of a trivial scenario are default constraints.

Observation 8 *Let ξ and η be two consistent scenarios with the same sequence of events. If ξ is trivial, then $\xi \cap \eta = \eta$ and $\xi \Cup \eta = \xi$.*

Lemma 5. *Let ξ and η be two consistent scenarios. If $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then neither ξ nor η is trivial.*

Proof. Assume ξ is trivial. Then, by Observation 8, $\xi \Cup \eta = \xi$.
Let $\zeta \in \mathcal{Z}(\xi, \eta)$. By Lemma 4, $[\![\zeta]\!] \cap [\![\xi]\!] = \emptyset$. So $[\![\zeta]\!] \cap [\![\xi \Cup \eta]\!] = \emptyset$. But this is contradictory to Lemma 3.

The consequence of Lemma 5 is that if $\mathcal{Z}(\xi, \eta)$ is not empty, then ξ and η must each have at least one *explicit* constraint.

Lemma 6. *Let ξ and η be two consistent scenarios. If $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then $\xi \not\subseteq \eta$ and $\eta \not\subseteq \xi$.*

Proof. Assume $\xi \subseteq \eta$ or $\eta \subseteq \xi$.
Let ζ be a scenario in $\mathcal{Z}(\xi, \eta)$. By Lemma 4, $[\![\zeta]\!] \cap [\![\xi]\!] = \emptyset$, and $[\![\zeta]\!] \cap [\![\eta]\!] = \emptyset$.
If $\eta \subseteq \xi$, then by Observation 7, $\xi \Cup \eta = \xi$. Then $[\![\zeta]\!] \cap [\![\xi \Cup \eta]\!] = \emptyset$. But this is contradictory to Lemma 3.
If $\xi \subseteq \eta$, then by a similar argument we can show a contradiction.

We will now examine ways of determining whether $\mathcal{Z}(\xi, \eta) \neq \emptyset$. This will be important for Theorem 3 at the end of this section.

Lemma 7. *Let $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ be two optimized scenarios. If $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then there must be $c_1 \in \mathcal{C}_1$ and $c_2 \in \mathcal{C}_2$, such that $c_1 \notin \mathcal{C}_2$ and $c_2 \notin \mathcal{C}_1$.*

Proof. By Lemma 5, ξ and η are not trivial. That is, $\mathcal{C}_1 \neq \emptyset$ and $\mathcal{C}_2 \neq \emptyset$. Assume that every constraint in $\mathcal{C}_1$ is also in $\mathcal{C}_2$, or every constraint in $\mathcal{C}_2$ is also in $\mathcal{C}_1$. Then $\eta \subseteq \xi$ or $\xi \subseteq \eta$. Then, by Lemma 6, $\mathcal{Z}(\xi, \eta) = \emptyset$. But this is a contradiction.

In the rest of the paper when we write $ij \neq kl$, we mean $i \neq k \vee j \neq l$.

Lemma 8. *Let $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ be two optimized scenarios of length n, such that $\xi \Cup \eta$ is defined and $\mathcal{Z}(\xi, \eta) \neq \emptyset$.*
If $\forall_{c_1 = \tau_{i,j} \sim a \in \mathcal{C}_1} \forall_{c_2 = \tau_{k,l} \sim b \in \mathcal{C}_2} (c_1 \in \mathcal{C}_2 \ \vee \ c_2 \in \mathcal{C}_1 \ \vee \ (i = k \wedge j = l))$, then there exists $0 \leq u < v < n$ such that every constraint in $\mathcal{C}_1 \setminus \mathcal{C}_2$ and every constraint in $\mathcal{C}_2 \setminus \mathcal{C}_1$ is between u and v.

Proof. By Lemma 7, $\mathcal{C}_1 \setminus \mathcal{C}_2 \neq \emptyset$ and $\mathcal{C}_2 \setminus \mathcal{C}_1 \neq \emptyset$.

Let c_1 be a constraint for some $\tau_{p,q}$ $(0 \leq p < q < n)$ in $\mathcal{C}_1 \setminus \mathcal{C}_2$. Then, by the assumption, every constraint in $\mathcal{C}_2$ is either between events p and q, or is also in $\mathcal{C}_1$. Similarly, every constraint in $\mathcal{C}_1$ is either between events p and q or is also in $\mathcal{C}_2$, so all the other constraints in $\mathcal{C}_1 \setminus \mathcal{C}_2$ (if any) must also be between events p and q.

Theorem 2. *Let $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ be two optimized scenarios of length n such that $\xi \Cup \eta$ is defined. If $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then there exists a constraint for some $\tau_{i,j}$ $(0 \leq i < j < n)$ in $\mathcal{C}_1$, which is not in $\mathcal{C}_2$, and there exists a constraint for some $\tau_{k,l}$ $(0 \leq k < l < n, ij \neq kl)$ in $\mathcal{C}_2$, which is not in $\mathcal{C}_1$.*

Proof. Assume the negation of the conclusion, that is,
$\forall_{c_1 = \tau_{i,j} \sim a \in \mathcal{C}_1} \forall_{c_2 = \tau_{k,l} \sim b \in \mathcal{C}_2} (c_1 \in \mathcal{C}_2 \ \vee \ c_2 \in \mathcal{C}_1 \ \vee \ (i = k \wedge j = l))$.

By Lemma **??**, the explicit constraints of ξ and η must be identical, except for those that are between a single pair of event numbers, say p and q.

In that case, for every $0 \leq i < j < n$, the intervals I^{ξ}_{ij} and I^{η}_{ij} both correspond to explicit constraints or both correspond to implicit constraints in ξ and η, respectively.

$\mathcal{Z}(\xi, \eta) \neq \emptyset$, so there exists some ζ in $\mathcal{Z}(\xi, \eta)$. By pt. 3 of Definition 7, there must exist $0 \leq u < v < n$ and $0 \leq w < z < n$, where $uv \neq wz$, such that
$I^{\xi}_{uv} \cap I^{\zeta}_{uv} = \emptyset$, $I^{\eta}_{uv} \cap I^{\zeta}_{uv} \neq \emptyset$, and $I^{\xi}_{wz} \cap I^{\zeta}_{wz} \neq \emptyset$, $I^{\eta}_{wz} \cap I^{\zeta}_{wz} = \emptyset$.

So $I^{\xi}_{uv} \neq I^{\eta}_{uv}$ and $I^{\xi}_{wz} \neq I^{\eta}_{wz}$. At least one of the intervals between u and v and between w and z must correspond to implicit constraints in *both* ξ and η. This is because there is only one pair of event numbers, p and q, that correspond to different explicit constraints in ξ and η.

We consider two cases:

1. Both I^{ξ}_{uv} and I^{η}_{wz} correspond to implicit constraints in ξ and η.
   Assume I^{ξ}_{uv} corresponds to some implicit constraint d in ξ. Because $I^{\xi}_{uv} \cap I^{\zeta}_{uv} = \emptyset$, constraint d is not satisfied by the behaviours in $[\![\zeta]\!]$. But then, by Observation 3, at least one explicit constraint $\alpha \in \mathcal{C}_1$ that is in the support of d must also not be satisfied by these behaviours.
   Similarly, assume I^{ξ}_{wz} corresponds to some implicit constraint e in η. Because $I^{\eta}_{wz} \cap I^{\zeta}_{wz} = \emptyset$, constraint e is not satisfied by the behaviours in $[\![\zeta]\!]$. By Observation 3 at least one explicit constraint $\beta \in \mathcal{C}_2$ $(\beta \neq \alpha)$ that is in the support of e must also not be satisfied by these behaviours. But in that case, there are two possibilities:
   (a) α is of the form $\tau_{p,q} \sim a$ and β is of the form $\tau_{p,q} \sim b$. Neither α nor β are satisfied by members of $[\![\zeta]\!]$, so $I^{\xi}_{pq} \cap I^{\zeta}_{pq} = \emptyset$, and $I^{\eta}_{pq} \cap I^{\zeta}_{pq} = \emptyset$. But then, $I^{\xi \Cup \eta}_{pq} \cap I^{\zeta}_{pq} = \emptyset$, which is in contradiction to pt. (2) of Definition 7.
   (b) One of α or β corresponds to $\tau_{p,q}$.
   Let α be of the form $\tau_{p,q} \sim a$ and β be of the form $\tau_{r,s} \sim b$, where $rs \neq pq$. Then β must also belong to $\mathcal{C}_1$. Therefore, β in $\mathcal{C}_1$ is also not satisfied by the behaviours in $[\![\zeta]\!]$. So, $I^{\xi}_{rs} \cap I^{\zeta}_{rs} = \emptyset$, and $I^{\eta}_{rs} \cap I^{\zeta}_{rs} = \emptyset$. But then, $I^{\xi \Cup \eta}_{rs} \cap I^{\zeta}_{rs} = \emptyset$, which is in contradiction to pt. (2) of Definition 7.
   If β is of the form $\tau_{p,q} \sim a$, we can show a contradiction by a similar argument.

2. I^{ξ}_{uv} corresponds to an explicit constraint in both ξ and η, while I^{ξ}_{wz} corresponds to an implicit constraint in both ξ and η. Then $uv = pq$.
   Let I^{ξ}_{pq} and I^{η}_{pq} correspond to explicit constraints d and d' in ξ and η, respectively. Because $I^{\xi}_{uv} \cap I^{\zeta}_{uv} = I^{\xi}_{pq} \cap I^{\zeta}_{pq} = \emptyset$, constraint d is not satisfied by the behaviours in $[\![\zeta]\!]$. But because $I^{\eta}_{uv} \cap I^{\zeta}_{uv} = I^{\eta}_{pq} \cap I^{\zeta}_{pq} \neq \emptyset$, constraint d' is satisfied by those behaviours.
   Let I^{ξ}_{wz} and I^{η}_{wz} correspond to implicit constraints e and e' in ξ and η respectively. Since $I^{\xi}_{wz} \cap I^{\zeta}_{wz} \neq \emptyset$, constraint e is satisfied by the behaviours in $[\![\zeta]\!]$, but e' is not satisfied by those behaviours, because $I^{\eta}_{wz} \cap I^{\zeta}_{wz} = \emptyset$. So e' is not a default constraint, and $e \neq e'$.
   By Observation 3 at least one explicit constraint $\beta \in \mathcal{C}_2$ that is in a support of e' must also not be satisfied by the members of $[\![\zeta]\!]$. β cannot be equal to d', because d' is satisfied by behaviours in $[\![\zeta]\!]$.
   Then β must also belong to $\mathcal{C}_1$. Therefore, β in $\mathcal{C}_1$ is also not satisfied by the behaviours in $[\![\zeta]\!]$. Let β correspond to some constraint between r and s. Then, $I^{\xi}_{rs} \cap I^{\zeta}_{rs} = \emptyset$, and $I^{\eta}_{rs} \cap I^{\zeta}_{rs} = \emptyset$. But then, $I^{\xi \Cup \eta}_{rs} \cap I^{\zeta}_{rs} = \emptyset$, which is in contradiction to pt. 2 of Definition 7.

Observation 9 *Let $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ be two optimized scenarios of length n such that $\xi \Cup \eta$ is defined. If there is no pair of constraints $\alpha = \tau_{i,j} \in \mathcal{C}_1 \setminus \mathcal{C}_2$ and $\beta = \tau_{k,l} \in \mathcal{C}_2 \setminus \mathcal{C}_1$, such that $(0 \leq i < j < n)$, $(0 \leq k < l < n)$ and $(ij \neq kl)$, then $\mathcal{Z}(\xi, \eta) = \emptyset$.*

Observation 9 is just a restatement of Theorem 2. It provides a sufficient condition for the non-existence of zigzagging behaviours: if $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ do not contain such an α and β, then $\mathcal{Z}(\xi, \eta) = \emptyset$. It is not clear whether the condition is also necessary.

Assume that $\xi \Cup \eta$ contains a pair of constraints α and β such that $\alpha = \tau_{i,j} \in \mathcal{C}_1 \setminus \mathcal{C}_2$, $\beta = \tau_{k,l} \in \mathcal{C}_2 \setminus \mathcal{C}_1$ and $(ij \neq kl)$. Consider $I_1 = I^{\xi}_{ij} \setminus I^{\eta}_{ij}$ and $I_2 = I^{\eta}_{kl} \setminus I^{\xi}_{kl}$. We construct a scenario $\zeta = (\mathcal{E}, \mathcal{C})$ where $\mathcal{C} = \{min\{I_1\} \leq \tau_{i,j},\ \tau_{i,j} \leq max\{I_1\},\ min\{I_2\} \leq \tau_{k,l},\ \tau_{k,l} \leq max\{I_2\}\}$. The behaviours in $[\![\zeta]\!]$ will belong neither to $[\![\xi]\!]$ nor to $[\![\eta]\!]$. But ζ might be inconsistent. However, this is quite easy to check. We must stabilize ζ's initial table, i.e., $\mathcal{D}^{\zeta}$. If the resulting table is valid, it is $\mathcal{D}^{\zeta}_{s}$, so we can check whether $\zeta \cap (\xi \Cup \eta)$ is defined and consistent. If so, then there exists at least one zigzagging behaviour in $[\![\zeta]\!]$: $\mathcal{Z}(\xi, \eta) \neq \emptyset$. If no zigzagging behaviour can be found for any such pair of constraints α and β, then $\mathcal{Z}(\xi, \eta) = \emptyset$.

Observation 10 *Let ξ and η be two scenarios such that $\xi \Cup \eta$ is defined. If the assumptions of Observation 9 are not satisfied and the process outlined in the preceding paragraph does not find a zigzagging behaviour, then $\mathcal{Z}(\xi, \eta) = \emptyset$.*

As an example consider scenarios ξ and η of Fig. 9 along with their stable distance tables once more. Observe that ξ and η are optimized: it is not possible to transform the set of explicit constraints of each to another set. By comparing the corresponding intervals of the distance tables, it is easy to see that ξ and η

are not subsets of each other. Moreover, $\xi \Cup \eta$ (shown on the left of Fig. 10) is defined (and therefore consistent).

As we mentioned before the union of the behaviours in $[\![\xi]\!]$ and $[\![\eta]\!]$ *cannot* be captured by their union: $\mathcal{Z}(\xi,\eta) \neq \emptyset$. In fact, ζ, shown on the right of Fig. 10, belongs to $\mathcal{Z}(\xi,\eta)$. Obviously, ζ is consistent: it allows all behaviours in which the time distance between a and b is at least 5, and between a and c is at most 6. Clearly, these behaviours are in $\xi \Cup \eta$, but are neither in $[\![\xi]\!]$, nor in $[\![\eta]\!]$. So in accordance with Observation 9 there must be a pair of constraints, one in ξ but not in η, and another one in η, but not in ξ, that are not between the same events. Indeed, the only explicit constraint of ξ (which is not in η) is between events 0 and 1, while η's only explicit constraint (which is not in ξ) is between events 0 and 2.

As another example consider the scenarios of Fig. 11 once more. The explicit constraints of ξ and η differ only on the constraints between one pair of events, namely, events 0 and 2, so by Theorem 2, $\mathcal{Z}(\xi,\eta) = \emptyset$.

Observation 11 *If $\mathcal{Z}(\xi,\eta) = \emptyset$, then every behaviour in $[\![\xi \Cup \eta]\!]$ must satisfy* all *the constraints in ξ, or* all *the constraints in η, or* all *the constraints in both ξ and η.*

Lemma 9. *Let ξ and η be two consistent scenarios with the same sequence of events, such that $\xi \Cup \eta$ is defined. If $[\![\xi \Cup \eta]\!] = [\![\xi]\!] \cup [\![\eta]\!]$, then $\mathcal{Z}(\xi,\eta) = \emptyset$.*

Proof. Assume $\mathcal{Z}(\xi,\eta) \neq \emptyset$. Let $\zeta \in \mathcal{Z}(\xi,\eta)$. By Lemma 3, $[\![\xi \Cup \eta]\!] \cap [\![\zeta]\!] \neq \emptyset$. Then $([\![\xi]\!] \cup [\![\eta]\!]) \cap [\![\zeta]\!] \neq \emptyset$. But then $[\![\xi]\!] \cap [\![\zeta]\!] \neq \emptyset$ or $[\![\eta]\!] \cap [\![\zeta]\!] \neq \emptyset$, which is contradictory to Lemma 4.

Lemma 10. *Let ξ and η be two consistent scenarios with the same sequence of events, such that $\xi \Cup \eta$ is defined. If $\mathcal{Z}(\xi,\eta) = \emptyset$, then $[\![\xi \Cup \eta]\!] = [\![\xi]\!] \cup [\![\eta]\!]$.*

Proof. By Lemma 2, $[\![\xi]\!] \cup [\![\eta]\!] \subseteq [\![\xi \Cup \eta]\!]$. Next, we show that $[\![\xi \Cup \eta]\!] \subseteq [\![\xi]\!] \cup [\![\eta]\!]$. Let $\mathcal{B} \in [\![\xi \Cup \eta]\!]$. Then
$\forall_{0 \le i < j < n}\ t^{\mathcal{B}}_{ij} \ge m^{\xi \Cup \eta}_{ij} = l^{\xi \Cup \eta}_{ij} = \min(m^{\xi}_{ij}, m^{\eta}_{ij})$, and
$\forall_{0 \le i < j < n}\ t^{\mathcal{B}}_{ij} \le M^{\xi \Cup \eta}_{ij} = h^{\xi \Cup \eta}_{ij} = \max(M^{\xi}_{ij}, M^{\eta}_{ij})$.
Without loss of generality assume that $m^{\xi}_{ij} \le m^{\eta}_{ij}$.
$\xi \Cup \eta$ is defined, so by Definition 6, $\forall_{0 \le i < j < n}\ I^{\xi}_{ij} \cap I^{\eta}_{ij} \neq \emptyset$.
Therefore $m^{\xi}_{ij} \le m^{\eta}_{ij} \le M^{\xi}_{ij} \le M^{\eta}_{ij}$ or $m^{\xi}_{ij} \le m^{\eta}_{ij} \le M^{\eta}_{ij} \le M^{\xi}_{ij}$.
If $m^{\xi}_{ij} \le m^{\eta}_{ij} \le M^{\xi}_{ij} \le M^{\eta}_{ij}$, then there are three cases:

1. $m^{\xi}_{ij} \le t^{\mathcal{B}}_{ij} \le m^{\eta}_{ij}$. Then $m^{\xi}_{ij} \le t^{\mathcal{B}}_{ij} \le M^{\xi}_{ij}$. So $t^{\mathcal{B}}_{ij} \in I^{\xi}_{ij}$, but $t^{\mathcal{B}}_{ij} \notin I^{\eta}_{ij}$.
2. $m^{\eta}_{ij} \le t^{\mathcal{B}}_{ij} \le M^{\xi}_{ij}$. Then $m^{\xi}_{ij} \le t^{\mathcal{B}}_{ij} \le M^{\xi}_{ij}$ and $m^{\eta}_{ij} \le t^{\mathcal{B}}_{ij} \le M^{\eta}_{ij}$. So $t^{\mathcal{B}}_{ij} \in I^{\xi}_{ij}$ and $t^{\mathcal{B}}_{ij} \in I^{\eta}_{ij}$.
3. $M^{\xi}_{ij} \le t^{\mathcal{B}}_{ij} \le M^{\eta}_{ij}$. Then $m^{\eta}_{ij} \le t^{\mathcal{B}}_{ij} \le M^{\eta}_{ij}$. So $t^{\mathcal{B}}_{ij} \in I^{\eta}_{ij}$, but $t^{\mathcal{B}}_{ij} \notin I^{\xi}_{ij}$.

If $m^{\xi}_{ij} \le m^{\eta}_{ij} \le M^{\eta}_{ij} \le M^{\xi}_{ij}$, then there are two additional cases:

4. $m^{\eta}_{ij} \le t^{\mathcal{B}}_{ij} \le M^{\eta}_{ij}$. Then $m^{\xi}_{ij} \le t^{\mathcal{B}}_{ij} \le M^{\xi}_{ij}$ and $m^{\eta}_{ij} \le t^{\mathcal{B}}_{ij} \le M^{\eta}_{ij}$. So $t^{\mathcal{B}}_{ij} \in I^{\xi}_{ij}$ and $t^{\mathcal{B}}_{ij} \in I^{\eta}_{ij}$.

5. $M^{\eta}_{ij} \leq t^{\mathcal{B}}_{ij} \leq M^{\xi}_{ij}$. Then $m^{\xi}_{ij} \leq t^{\mathcal{B}}_{ij} \leq M^{\xi}_{ij}$. So $t^{\mathcal{B}}_{ij} \in I^{\xi}_{ij}$, but $t^{\mathcal{B}}_{ij} \notin I^{\eta}_{ij}$.

$\mathcal{Z}(\xi, \eta) = \emptyset$. So for a given $\mathcal{B}$, if $t^{\mathcal{B}}_{ij} \in I^{\xi}_{ij}$, but $t^{\mathcal{B}}_{ij} \notin I^{\eta}_{ij}$, it is not possible to have $t^{\mathcal{B}}_{kl} \notin I^{\xi}_{kl}$, and $t^{\mathcal{B}}_{kl} \in I^{\eta}_{kl}$, for some $kl \neq ij$. Therefore, if $t^{\mathcal{B}}_{ij}$ satisfies the requirements in case (1) or case (5), there is no $kl \neq ij$ for which the requirements in (3) are satisfied, and vice versa. Then either all $t^{\mathcal{B}}_{ij}$ (for any $0 \leq i \leq j < n$) of $\mathcal{B}$
(a) satisfy the requirements in case (1) or case (5), or
(b) satisfy the requirements in case (2) or case (4), or
(c) satisfy the requirements in case (3).
If (a), then $\mathcal{B} \in [\![\xi]\!]$. If (b), then $\mathcal{B} \in [\![\xi]\!]$ and $\mathcal{B} \in [\![\eta]\!]$. If (c), then $\mathcal{B} \in [\![\eta]\!]$.
So $\mathcal{B} \in [\![\xi]\!] \cup [\![\eta]\!]$. Therefore $[\![\xi \Cup \eta]\!] \subseteq [\![\xi]\!] \cup [\![\eta]\!]$.

Theorem 3. *Let ξ and η be two consistent scenarios with the same sequence of events, such that $\xi \Cup \eta$ is defined. Then $[\![\xi \Cup \eta]\!] = [\![\xi]\!] \cup [\![\eta]\!]$ iff $\mathcal{Z}(\xi, \eta) = \emptyset$.*

Proof. This is a direct consequence of Lemma 8 and Lemma 9.

5 Conclusions and related work

We introduce subsumption, intersection and union operations for timed scenarios. We formally define these operations and their semantics.

Given two consistent scenarios ξ and η, ξ is subsumed by η, i.e., $\xi \subseteq \eta$, iff $[\![\xi]\!] \subseteq [\![\eta]\!]$, that is, the set of behaviours allowed by ξ is a subset of that for η. We provide a constructive way of verifying whether that is the case (Definition 4 and Observation 4).

Next, we show that for two consistent scenarios ξ and η, if the set of behaviours allowed by both ξ and η is not empty, then the set can be captured by a single scenario, namely the intersection of ξ and η, i.e., $\xi \cap \eta$. Definition 5 shows how to construct $\xi \cap \eta$ and Theorem 1 proves its semantic properties.

Then we turn our attention to the union of scenarios. Given two consistent scenarios ξ and η, we want to know whether there exists a single scenario that would express the set of behaviours that are allowed by ξ or η, i.e., the union of the behaviours allowed by ξ and η. We constructively define $\xi \Cup \eta$ (Definition 6) and show that it captures the expected properties of union only under certain conditions: in particular, when it does not contain "zigzagging" behaviours (Theorem 3). Intuitively, zigzagging behaviours satisfy all the constraints of $\xi \Cup \eta$ but do not satisfy some of the constraints in ξ and some of the constraints in η. In other words, they belong neither to $[\![\xi]\!]$, nor to $[\![\eta]\!]$, even though they belong to $[\![\xi \Cup \eta]\!]$. We formalize zigzagging behaviours, and show that such behaviours are possible *only* when there is a certain relationship between the constraints of ξ and η (Theorem 2). This provides us with a *syntactic* criterion for determining whether $\xi \Cup \eta$ does indeed have the required semantic properties of union.

To the best of our knowledge this is the first attempt at developing such operations for timed scenarios. These operations are directly relevant to the problem of synthesizing timed automata with minimal number of clocks from a set of scenarios (see Sec. 3 for a brief discussion), which we will address in a forthcoming paper.

A detailed comparison of timed scenarios with other related work, in particular with Difference Bounds Matrices (DBMs), can be found in our earlier work [9, 13].

Inclusion and intersection operations have been defined for DBMs [19]. However, our subsumption and intersection operations, developed from first principles for timed scenarios, are naturally very different.

Union of DBMs has been handled by convex hulls, leading to an approximation algorithm [20] and a safe abstraction [21]: the union of two zones (represented by DBMs) is generally a non-convex set, and therefore cannot be represented by a zone, i.e., a DBM.

Another method for checking whether the union of two DBMs is itself a DBM has been developed using convex hulls together with Clock Difference Diagrams (CDDs) [22].

Our union operation, with well-defined semantics, is defined in a different and straightforward way using the stable distance tables. An interesting property of our union is that the initial distance table corresponding to the union of two scenarios is already *stable*: there is no need for stabilization (Observation 5). More importantly, no approximation is required, thanks to our syntactic criterion used for the existance of union.

References

1. S. Somé, R. Dssouli, J. Vaucher, From Scenarios to Timed Automata: Building Specifications from Users Requirements, in: Proceedings of the Second Asia Pacific Software Engineering Conference, APSEC '95, IEEE Computer Society, pp. 48–57.
2. P. Chandrasekaran, M. Mukund, Matching scenarios with timing constraints, in: E. Asarin, P. Bouyer (Eds.), Formal Modeling and Analysis of Timed Systems, Springer, Berlin, Heidelberg, 2006, pp. 98–112.
3. D. Harel, H. Kugler, A. Pnueli, Synthesis Revisited: Generating Statechart Models from Scenario-Based Requirements, Springer Berlin Heidelberg, 2005, pp. 309–324.
4. S. Uchitel, J. Kramer, J. Magee, Synthesis of Behavioral Models from Scenarios, IEEE Trans. Softw. Eng. 29 (2) (2003) 99–115.
5. S. Akshay, M. Mukund, K. N. Kumar, Checking Coverage for Infinite Collections of Timed Scenarios, in: CONCUR 2007 - Concurrency Theory, 18th International Conference, CONCUR 2007, Proceedings, pp. 181–196.
6. B. Bollig, J. Katoen, C. Kern, M. Leucker, Replaying Play In and Play Out: Synthesis of Design Models from Scenarios by Learning, in: Proceedings of the 13th International Conference on Tools and Algorithms for the Construction and Analysis of Systems, TACAS, 2007, pp. 435–450.
7. R. Alur, M. Martin, M. Raghothaman, C. Stergiou, S. Tripakis, A. Udupa, Synthesizing Finite-State Protocols from Scenarios and Requirements, in: E. Yahav (Ed.), Hardware and Software: Verification and Testing, Springer International Publishing, Cham, 2014, pp. 75–91.
8. C. L. Heitmeyer, M. Pickett, E. I. Leonard, M. M. Archer, I. Ray, D. W. Aha, J. G. Trafton, Building high assurance human-centric decision systems, Autom. Softw. Eng. 22 (2) (2015) 159–197. doi:10.1007/s10515-014-0157-z.

9. N. Saeedloei, F. Kluźniak, From Scenarios to Timed Automata, in: Formal Methods: Foundations and Applications - 20th Brazilian Symposium, SBMF 2017, Proceedings, pp. 33–51.
10. N. Saeedloei, F. Kluźniak, Timed Scenarios: Consistency, Equivalence and Optimization, in: Formal Methods: Foundations and Applications - 21st Brazilian Symposium, SBMF 2018, Proceedings, pp. 215–233.
11. R. Alur, D. L. Dill, A theory of timed automata, Theor. Comput. Sci. 126 (2) (1994) 183–235.
12. R. Alur, P. Madhusudan, Decision Problems for Timed Automata: A Survey, in: Formal Methods for the Design of Real-Time Systems, International School on Formal Methods for the Design of Computer, Communication and Software Systems, SFM-RT, Revised Lectures, 2004, pp. 1–24.
13. N. Saeedloei, F. Kluźniak, Optimization of timed scenarios, in: G. Carvalho, V. Stolz (Eds.), Formal Methods: Foundations and Applications - 23rd Brazilian Symposium, SBMF 2020, Ouro Preto, Brazil, November 25-27, 2020, Proceedings, Vol. 12475 of Lecture Notes in Computer Science, Springer, 2020, pp. 119–136.
14. N. Saeedloei, F. Kluzniak, Minimization of the number of clocks for timed scenarios, in: S. Campos, M. Minea (Eds.), Formal Methods: Foundations and Applications - 24th Brazilian Symposium, SBMF 2021 Proceedings, Vol. 13130 of Lecture Notes in Computer Science, Springer, 2021, pp. 122–139.
15. P. A. Abdulla, J. Deneux, J. Ouaknine, J. Worrell, Decidability and complexity results for timed automata via channel machines, in: L. Caires, G. F. Italiano, L. Monteiro, C. Palamidessi, M. Yung (Eds.), Automata, Languages and Programming, Springer, Berlin, Heidelberg, 2005, pp. 1089–1101.
16. C. Baier, N. Bertrand, P. Bouyer, T. Brihaye, When are timed automata determinizable?, in: S. Albers, A. Marchetti-Spaccamela, Y. Matias, S. Nikoletseas, W. Thomas (Eds.), Automata, Languages and Programming, Springer, Berlin, Heidelberg, 2009, pp. 43–54.
17. D. L. Dill, Timing assumptions and verification of finite-state concurrent systems, in: Proceedings of the International Workshop on Automatic Verification Methods for Finite State Systems, Springer-Verlag New York, Inc., 1990, pp. 197–212.
18. K. G. Larsen, F. Larsson, P. Pettersson, W. Yi, Efficient verification of real-time systems: Compact data structure and state-space reduction,, in: 18th IEEE Real-Time Systems Symposium, 1997, pp. 14–24.
19. J. Bengtsson, W. Yi, Timed Automata: Semantics, Algorithms and Tools, Springer Berlin Heidelberg, Berlin, Heidelberg, 2004, pp. 87–124.
20. F. Balarin, Approximate reachability analysis of timed automata, in: 17th IEEE Real-Time Systems Symposium, 1996, pp. 52–61. doi:10.1109/REAL.1996.563700.
21. C. Daws, S. Tripakis, Model Checking of Real-Time Reachability Properties Using Abstractions, in: Proceedings of the Fourth International Conference on Tools and Algorithms for the Construction and Analysis of Systems, Vol. 1384 of Lecture Notes in Computer Science, Springer-Verlag, 1998, pp. 313–329. doi:10.1007/BFb0054180.
22. G. Behrmann, K. G. Larsen, J. Pearson, C. Weise, W. Yi, Efficient timed reachability analysis using clock difference diagrams, in: N. Halbwachs, D. Peled (Eds.), Computer Aided Verification, Springer Berlin Heidelberg, Berlin, Heidelberg, 1999, pp. 341–353.