

On the Existence of Unions of Timed Scenarios

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Abstract. In earlier work it was shown that, given two consistent timed scenarios, there might not exist a scenario whose semantics is the union of those of the two. A sufficient condition for the non-existence of such a union was also identified. In this paper we report on a comprehensive study of the union operation for scenarios. We identify a new sufficient condition that provides a syntactic criterion for the existence of a scenario that is the union of two given scenarios. We also prove that the condition is necessary.

1 Introduction

Using scenarios for specifying complex systems (including real time systems and distributed systems [1, 2]), and synthesizing formal models of systems from scenarios have been active areas of research for several decades [3–10].

In our earlier work [11] we developed a formal, yet simple notation for timed scenarios. Intuitively, a scenario is a sequence of events along with a set of constraints between the times of these events, which can be used to specify the partial behaviours of a system or a component of a system (see Sec. 2 for more details). We defined the semantics of a timed scenario as the set of all behaviours that are “allowed” by the scenario. All such behaviours must satisfy the same set of constraints. We developed the notion of “stable distance table”, as a *canonical* representation of the set of constraints of a scenario: it captures the tightest set of constraints of the scenario. We used stable distance tables as the foundation for various algorithms for determining the consistency and equivalency of scenarios, as well as for optimizing scenarios [11, 12]. More recently, we used them for developing the notions of intersection, union and subsumption for scenarios [13]. In particular, we studied the conditions under which the union of the behaviours allowed by two scenarios can be represented by a single scenario, and presented a sufficient condition for the non-existence of the union [13]. The problem of whether the condition is also necessary was left open [13].

The operations on scenarios—the union, in particular—are directly relevant to the problem of synthesizing timed automata from a set of scenarios [14]. Assume a set Ξ of scenarios includes ξ and η such that $\gamma = \xi \cup \eta$. Then replacing both ξ and η in Ξ with γ , before synthesis, always results in fewer locations, and often in fewer clocks in the synthesized automaton [14].

These operations are, in general, also relevant to model-checking timed automata [15]. Model-checking [16] is an automatic technique based on exhaustive

exploration of the reachable state space of a system: given a model of a system and some property p , the goal is to determine whether there exists a state reachable from the initial state in which p holds. A major obstacle to using model-checking in practice is the state explosion problem. For real-time systems modeled as timed automata the state space is infinite due to clock variables, therefore a complete search of the state space is impossible. However, the reachability problem for timed automata is decidable [15], thanks to various abstraction techniques that have been developed to overcome the state space explosion problem [17]. One such abstraction is using clock zones, as symbolic representations of the states of timed automata. A clock zone is a set of constraints, each of which puts a bound on the difference between the values of two clocks. Each location of a timed automaton is associated with a zone, and the reachability analysis involves manipulating clock zones along various paths of the automaton. A location, in general, can be associated with more than one zone, for example when it can be reached via different paths. In that case, the union of all these zones must be taken, in order to get the zone corresponding to the location. However, the union of zones, in general, cannot be represented by a zone, and therefore, it is often approximated [18, 17]. One such approximation is obtained by taking the convex hull of the zones [17].

Various data structures for representing zones, and a variety of reachability algorithms utilizing these data structures have been introduced [19, 17, 20]. Examples include using difference bound matrices (DBMs) [21], binary decision diagrams (BDDs) [18], clock difference diagrams (CDDs) [19] and Numerical decision diagrams (NDDs) [22, 20].

As part of an earlier work [12, 23] we showed how—in the case of a timed automaton that corresponds to a single timed scenario—our distance tables can be used to represent time zones similarly to DBMs. So the next natural step for us is to investigate the necessary conditions under which it is possible to form the union of two timed scenarios represented by two distance tables. More specifically, we make the following contributions:

- We introduce the notion of z_pairs between two different constraints α and β of two scenarios ξ and η , respectively. Intuitively, α and β form a z_pair , if there exist behaviours whose times do not satisfy constraints α and β , but satisfy the constraints of the “quasi-union” of ξ and η (see Sec. 2.1 for details). That is, they do not belong to the semantics of either ξ or η , but belong to the semantics of the “quasi-union” of ξ and η . We call such behaviours “zigzagging behaviours”. We show that if there is no z_pair between scenarios ξ and η , then $\xi \cup \eta$ exists. This will be a new sufficient condition for the existence of the union of two scenarios, but stronger than our previous condition [13].
- We prove that the existence of z_pairs is also a necessary condition for the existence of zigzagging behaviours, and therefore the non-existence of the union of ξ and η . This will address the problem that was previously left open [13].

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Fig. 1. ξ , its initial table and its stable table

2 Timed scenarios

This section briefly recounts our earlier work [11–13].

Let Σ be a finite set of symbols called *events*. A *behaviour*¹ over Σ is a sequence $(e_0, t_0)(e_1, t_1)(e_2, t_2) \dots$, such that $e_i \in \Sigma$, $t_i \in \mathbb{R}^{\geq 0}$ and $t_{i-1} \leq t_i$ for $i \in \{1, 2, \dots\}$. For a finite behaviour $\mathcal{B} = (e_0, t_0)(e_1, t_1) \dots (e_{n-1}, t_{n-1})$ of length n , and for any $0 \leq i < j < n$, the *distance*, in time units, of event j from event i in \mathcal{B} is denoted by $t_{ij}^{\mathcal{B}}$. That is, $t_{ij}^{\mathcal{B}} = t_j - t_i$.

A *timed scenario* (*scenario* for short) of length $n \in \mathbb{N}$ over Σ is a pair $(\mathcal{E}, \mathcal{C})$, where $\mathcal{E} = e_0 e_1 \dots e_{n-1}$ is a sequence of events, and $\mathcal{C} \subset \Phi(n)$ is a finite set of constraints. Each constraint in $\Phi(n)$ is of the form $b \sim a$, where b is the symbol $\tau_{i,j}$ (for some integers $0 \leq i < j < n$), $\sim \in \{\leq, \geq\}$ and a is a constant in the set of rational numbers, \mathbb{Q} . The interpretation is that $\tau_{i,j}$ is the time distance between the i -th and the j -th events in the behaviours described by a scenario. The constraints $\tau_{i,j} \geq 0$ and $\tau_{i,j} \leq \infty$ are called *default constraints*.

A behaviour $\mathcal{B} = (e_0, t_0)(e_1, t_1) \dots (e_{n-1}, t_{n-1})$ over Σ is *allowed* by scenario $\xi = (\mathcal{E}, \mathcal{C})$ iff $\mathcal{E} = e_0 \dots e_{n-1}$ and every $\tau_{i,j} \sim a$ in \mathcal{C} evaluates to true after $\tau_{i,j}$ is replaced by $t_{ij}^{\mathcal{B}}$. The *semantics* of scenario ξ , denoted by $\llbracket \xi \rrbracket$, is the set of behaviours that are allowed by ξ .

Fig. 1 shows the “external representation” of scenario $\xi = (abc, \{\tau_{0,1} \geq 3, \tau_{0,2} \leq 7\})$, $\llbracket \xi \rrbracket = \{(a, t_0)(b, t_1)(c, t_2) \mid t_0 \leq t_1 \leq t_2 \wedge t_1 - t_0 \geq 3 \wedge t_2 - t_0 \leq 7\}$.

A scenario ξ is *consistent* iff $\llbracket \xi \rrbracket \neq \emptyset$. It is *inconsistent* iff $\llbracket \xi \rrbracket = \emptyset$.

For a consistent scenario ξ of length n , and for $0 \leq i < j < n$, $m_{ij}^{\xi} = \min\{t_{ij}^{\mathcal{B}} \mid \mathcal{B} \in \llbracket \xi \rrbracket\}$ and $M_{ij}^{\xi} = \max\{t_{ij}^{\mathcal{B}} \mid \mathcal{B} \in \llbracket \xi \rrbracket\}$. The absence of an upper bound for some i and j will be denoted by $M_{ij}^{\xi} = \infty$. We will write just m_{ij} and M_{ij} when ξ is understood. For any behaviour \mathcal{B} in $\llbracket \xi \rrbracket$, $0 \leq m_{ij} \leq t_{ij}^{\mathcal{B}} \leq M_{ij} \leq \infty$.

For a consistent scenario ξ of length n , and for any $0 \leq i < j < k < n$ the following inequations hold [11]:

$$m_{ij} + m_{jk} \leq m_{ik} \leq \left\{ \begin{array}{l} m_{ij} + M_{jk} \\ M_{ij} + m_{jk} \end{array} \right\} \leq M_{ik} \leq M_{ij} + M_{jk} \quad (1)$$

Let $\xi = (\mathcal{E}, \mathcal{C})$ be a scenario of length n , such that, for any $0 \leq i < j < n$, \mathcal{C} contains at most one constraint of the form $\tau_{i,j} \geq c$ and at most one of the form $\tau_{i,j} \leq c$. A *distance table* for ξ is a representation of \mathcal{C} in the form of a triangular matrix \mathcal{D}^{ξ} . For $0 \leq i < j < n$, $\mathcal{D}^{\xi}[i, j] = (l_{ij}^{\xi}, h_{ij}^{\xi})$, where l_{ij}^{ξ} and h_{ij}^{ξ} are rational numbers. If $\tau_{i,j} \geq c \in \mathcal{C}$ then $l_{ij}^{\xi} = c$, otherwise $l_{ij}^{\xi} = 0$; if $\tau_{i,j} \leq c \in \mathcal{C}$ then

¹ The notion of “behaviour” is equivalent to that of Alur’s “timed word” [15].

$h_{ij}^\xi = c$, otherwise $h_{ij}^\xi = \infty$. We will write just l_{ij} and h_{ij} when ξ is understood. A distance table for ξ of Fig. 1 is shown in the middle of the figure.

A distance table for a scenario of size n is *valid* iff $l_{ij} \leq h_{ij}$, for all $0 \leq i < j < n$. A table that is not valid is *invalid*. If \mathcal{D}^ξ is invalid, then ξ is inconsistent.

A valid distance table for a scenario of size n is *stable* iff, for all $0 \leq i < j < k < n$, the inequations in (1) hold when m_{ij}, m_{jk}, m_{ik} are replaced by l_{ij}, l_{jk}, l_{ik} and M_{ij}, M_{jk}, M_{ik} are replaced by h_{ij}, h_{jk}, h_{ik} . If \mathcal{D}^ξ is stable then ξ is consistent. The table of a consistent scenario ξ can be stabilized by applying the following six rules:

$$\begin{aligned} l_{ij} + l_{jk} > l_{ik} &\longrightarrow l_{ik} := l_{ij} + l_{jk} & l_{ik} > l_{ij} + h_{jk} &\longrightarrow l_{ij} := l_{ik} - h_{jk} \\ l_{ik} > h_{ij} + l_{jk} &\longrightarrow l_{jk} := l_{ik} - h_{ij} & l_{ij} + h_{jk} > h_{ik} &\longrightarrow h_{jk} := h_{ik} - l_{ij} \\ h_{ij} + l_{jk} > h_{ik} &\longrightarrow h_{ij} := h_{ik} - l_{jk} & h_{ik} > h_{ij} + h_{jk} &\longrightarrow h_{ik} := h_{ij} + h_{jk} \end{aligned}$$

At least one rule is applicable if and only if some inequation in (1) does not hold. The purpose of each rule is to tighten a constraint (i.e., increase a lower bound or decrease an upper bound) just enough to establish a particular inequation. If none of the rules is applicable, and the table is valid, then it is stable. The table on the right of Fig. 1 is the stable distance table of ξ in that figure.

If ξ is a scenario of length n and \mathcal{D}_s^ξ is its stable table, then for every $0 \leq i < j < n$, $l_{ij}^\xi = m_{ij}^\xi$ and $h_{ij}^\xi = M_{ij}^\xi$, that is, $\mathcal{D}_s^\xi[i, j] = (m_{ij}^\xi, M_{ij}^\xi)$. Intuitively, $\mathcal{D}_s^\xi[i, j]$ corresponds to a pair of constraints: the time distance between events i and j in every behaviour in $\llbracket \xi \rrbracket$ must be at least m_{ij}^ξ , and at most M_{ij}^ξ . We use $\mathcal{C}(\mathcal{D}_s^\xi)$ to denote the set of constraints represented by \mathcal{D}_s^ξ . All the constraints in $\mathcal{C}(\mathcal{D}_s^\xi)$ are as *tight* as possible. Moreover, $\mathcal{C}(\mathcal{D}_s^\xi)$ includes all the constraints that are implied [12] by the initial set of constraints.

A scenario ξ can be transformed to an equivalent scenario η by *optimizing* its set of constraints [23], so that η has a minimal set of constraints². We call η the *optimized* form of ξ . For an optimized scenario $\xi = (\mathcal{E}, \mathcal{C})$ the members of \mathcal{C} will be referred to as *explicit constraints*. We know that \mathcal{D}_s^ξ also includes *implicit* constraints: default constraints and constraints that are implied by \mathcal{C} .

The stable table of Fig. 1 represents the set of constraints $\{\tau_{0,1} \geq 3, \tau_{0,1} \leq 7, \tau_{0,2} \geq 3, \tau_{0,2} \leq 7, \tau_{1,2} \geq 0, \tau_{1,2} \leq 4\}$. Observe that, for example, $\mathcal{D}_s^\xi[1, 2] = (0, 4)$: $m_{12}^\xi = 0$, which corresponds to a default constraint, and $M_{12}^\xi = 4$, which corresponds to an implied constraint. The set of explicit constraints of ξ , which is already in optimized form, is $\{\tau_{0,1} \geq 3, \tau_{0,2} \leq 7\}$, while $\{\tau_{0,1} \leq 7, \tau_{0,2} \geq 3, \tau_{1,2} \leq 4\}$ is the set of implicit constraints.

The *interval* I_{ij}^ξ is $\{a \in \mathbb{Q} \mid m_{ij}^\xi \leq a \leq M_{ij}^\xi\}$ where $\mathcal{D}_s^\xi[i, j] = (m_{ij}^\xi, M_{ij}^\xi)$. Intuitively, I_{ij}^ξ specifies the set of all the possible values of t_{ij} that can appear in the behaviours allowed by ξ . We will write just I_{ij} when ξ is understood.

If $i = j$, then by definition, $t_{ii} = 0$, so $m_{ii} = 0$, $M_{ii} = 0$ and $I_{ii} = [0, 0]$.

² The aim of optimization which begins with the construction of a stable distance table, is to obtain a minimal set of constraints, i.e., removal of any of these constraints would change the semantics.

η	$\begin{array}{l} 0 : a ; \\ 1 : b ; \\ 2 : c \{ \tau_{1,2} \geq 4 \} . \end{array}$	$\xi \uplus \eta$	$\begin{array}{l} 0 : a ; \\ 1 : b ; \\ 2 : c \{ \tau_{0,2} \geq 3 \} . \end{array}$	ζ	$\begin{array}{l} 0 : a ; \\ 1 : b \{ \tau_{0,1} \geq 2, \tau_{0,1} \leq 2 \}; \\ 2 : c \{ \tau_{1,2} \geq 3, \tau_{1,2} \leq 3 \} . \end{array}$																																				
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Fig. 2. η and its stable table **Fig. 3.** The combination of ξ of Fig. 1 and η of Fig. 2, and ζ with its stable table

2.1 Combination of two timed scenarios

If ξ and η are two consistent scenarios of length n with the same sequences of events, \mathcal{E} , such that $\forall_{0 \leq i < j < n} I_{ij}^\xi \cap I_{ij}^\eta \neq \emptyset$, then the *combination* (or the “quasi-union”) of ξ and η , denoted by $\xi \uplus \eta$, is *defined*. In that case, $\xi \uplus \eta$ is a scenario whose sequence of events is \mathcal{E} and whose constraints are given by $\mathcal{D}^{\xi \uplus \eta}$, where $\mathcal{D}^{\xi \uplus \eta}[i, j] = (\min(m_{ij}^\xi, m_{ij}^\eta), \max(M_{ij}^\xi, M_{ij}^\eta))$.

If $\xi \uplus \eta$ is defined, then $\llbracket \xi \rrbracket \cup \llbracket \eta \rrbracket \subseteq \llbracket \xi \uplus \eta \rrbracket$. But, in general, $\llbracket \xi \uplus \eta \rrbracket \not\subseteq \llbracket \xi \rrbracket \cup \llbracket \eta \rrbracket$. This is because table $\mathcal{D}_s^{\xi \uplus \eta}$ allows all the behaviours in $\llbracket \xi \rrbracket \cup \llbracket \eta \rrbracket$, but there is a possibility that it may also allow some extra behaviours, namely those that satisfy all the constraints of the combination, but do not satisfy some of the constraints in ξ and some of the constraints in η . That is, $\llbracket \xi \uplus \eta \rrbracket = \llbracket \xi \rrbracket \cup \llbracket \eta \rrbracket \cup \mathcal{Z}(\xi, \eta)$, where $\llbracket \xi \rrbracket \cap \mathcal{Z}(\xi, \eta) = \emptyset$ and $\llbracket \eta \rrbracket \cap \mathcal{Z}(\xi, \eta) = \emptyset$. We call members of $\mathcal{Z}(\xi, \eta)$ *zigzagging* behaviours.

As an example consider scenarios ξ of Fig. 1 and η of Fig. 2. Fig. 3 shows $\xi \uplus \eta$ along with its stable table. Scenario ζ of Fig. 3 represents a set of behaviours in which the time distance between events a and b is exactly 2, and between events b and c is exactly 3 units of time. There is no behaviour in the semantics of ζ that is allowed by either ξ or η , yet $\llbracket \zeta \rrbracket \subset \llbracket \xi \uplus \eta \rrbracket$. That is, all behaviours in $\llbracket \zeta \rrbracket$ belong to $\mathcal{Z}(\xi, \eta)$. *This indicates that the union of the sets of behaviours allowed by ξ and η cannot be represented by a single scenario.*

Given two consistent scenarios ξ and η of length n with the same sequence of events, \mathcal{E} , such that $\xi \uplus \eta$ is defined, behaviour \mathcal{B}^z with the sequence of events \mathcal{E} belongs to $\mathcal{Z}(\xi, \eta)$ iff (1) For every $0 \leq i < j < n$, $t_{ij}^{\mathcal{B}^z} \in I_{ij}^{\xi \uplus \eta}$, (2) There exist $0 \leq i < j < n$ and $0 \leq k < l < n$ such that (a) $i \neq k$ or $j \neq l$, (b) $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^\xi$, $t_{ij}^{\mathcal{B}^z} \in I_{ij}^\eta$, and (c) $t_{kl}^{\mathcal{B}^z} \in I_{kl}^\xi$, $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^\eta$.

There is a sufficient condition for the non-existence of zigzagging behaviours [13]: if $\mathcal{Z}(\xi, \eta) \neq \emptyset$ then ξ and η each has at least one explicit constraint, such that the constraints are not between the same events i and j in both ξ and η .

Given ξ and η , if $\mathcal{Z}(\xi, \eta) = \emptyset$, $\xi \uplus \eta$ becomes their *union*, denoted by $\xi \cup \eta$. Then, scenario $\xi \cup \eta$ captures the union of the sets of behaviours allowed by ξ , η or both: $\llbracket \xi \cup \eta \rrbracket = \llbracket \xi \rrbracket \cup \llbracket \eta \rrbracket$. Fig. 4 shows two scenarios and their union.

ξ	η	$\xi \cup \eta$																																																
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Fig. 4. Two scenarios and their union

3 Construction of zigzagging behaviours

In this section we develop a method for constructing zigzagging behaviours, along with the necessary definitions and theoretical foundations.

Definition 1 is a generalization of our previously-introduced notion of compatibility of a timed sequence with a stable distance table [11].

Definition 1. Let ξ be a scenario of size n and \mathcal{D}_s^ξ be its stable distance table. Let $0 \leq k < n$ and let $\mathcal{J} = \{b_0, b_1, \dots, b_k\} \subseteq \{i \in \mathbb{N} \mid 0 \leq i < n\}$, where $b_0 < b_1 < \dots < b_k$. The sequence of real numbers $S = t_{b_0} t_{b_1} \dots t_{b_k}$ is compatible with \mathcal{D}_s^ξ iff the following three conditions hold:

1. $b_0 > 0 \Rightarrow t_{b_0} \geq m_{0b_0}$,
2. $\forall i, l \in \mathcal{J} (i < l \Rightarrow t_i \leq t_l)$ and
3. $\forall i, l \in \mathcal{J} (i < l \Rightarrow t_{il} \in I_{il}^\xi)$.

Notice that if $k = 0$ then the sequence consists of one item. Conditions 2 and 3 are then trivially true. An empty sequence is not compatible with any stable table.

The sequence $t_1 t_2$, where $t_1 = 3$ and $t_2 = 4$, is compatible with the stable table for scenario ξ of Fig. 4: $t_1 \geq m_{01} = 0$, $t_1 \leq t_2$ and $t_{12} = t_2 - t_1 = 1 \in I_{12}^\xi$.

Observation 1 Let ξ be a scenario of size n , let \mathcal{D}_s^ξ be its stable distance table and let $e_0 e_1 \dots e_{n-1}$ be the events of ξ . The sequence $S = t_0 t_1 \dots t_{n-1}$ is compatible with \mathcal{D}_s^ξ iff $(e_0, t_0)(e_1, t_1) \dots (e_{n-1}, t_{n-1}) \in \llbracket \xi \rrbracket$.

Intuitively, a sequence that is compatible with a stable table of a scenario is the sequence of time values shared by the members of a set of behaviours allowed by the scenario. If the sequence has n elements, then the set is a singleton, i.e., the sequence represents one “complete” behaviour. If the sequence has less than n elements, it can be thought of as a behaviour that is only partially specified. Such a “partial” behaviour can always be extended to a complete behaviour, as shown in Theorem 1.

Theorem 1. *Let ξ be a scenario of size n and \mathcal{D}_s^ξ be its stable distance table. Let $\mathcal{J} = \{b_0, b_1, \dots, b_k\} \subseteq \{i \in \mathbb{N} \mid 0 \leq i < n\}$, $k \geq 0$, $b_0 < b_1 < \dots < b_k$ and let $S = t_{b_0}t_{b_1} \dots t_{b_k}$ be compatible with \mathcal{D}_s^ξ . Then, for any $r \in \{i \in \mathbb{N} \mid 0 \leq i < n\} \setminus \mathcal{J}$, there exists a real number t_r such that*

1. *if $b_k < r < n$, then $t_{b_0} \dots t_{b_k} \dots t_r$ is compatible with \mathcal{D}_s^ξ ;*
2. *if $b_0 < r < b_k$, then $t_{b_0} \dots t_r \dots t_{b_k}$ is compatible with \mathcal{D}_s^ξ ;*
3. *if $0 \leq r < b_0$, then $t_r \dots t_{b_0} \dots t_{b_k}$ is compatible with \mathcal{D}_s^ξ .*

Proof. (Fig. 5 illustrates the three cases of the theorem.) We show the proof for Case 1. The complete proof is presented in the longer version of the paper [24].

Case 1: If $k = 0$, then let $i = b_0$ and $b_0 < r$. We must show that there exists a real number t_r such that $t_i \leq t_r$ and $m_{ir} \leq t_{ir} \leq M_{ir}$. But because $m_{ir} \leq M_{ir}$, it is always possible to find a t_{ir} that satisfies the inequations.

Let $k > 0$, $b_k < r$ and let $i = b_p$ and $j = b_q$, where $0 \leq p < q \leq k$. We must show that there exists a real number t_r such that $t_i \leq t_j \leq t_r$ and

$$m_{jr} \leq t_{jr} \leq M_{jr} \quad (2) \quad m_{ir} \leq t_{ir} \leq M_{ir} \quad (3)$$

Inequation (2) is equivalent to $m_{jr} \leq t_{ir} - t_{ij} \leq M_{jr}$, which is equivalent to

$$t_{ij} + m_{jr} \leq t_{ir} \leq t_{ij} + M_{jr} \quad (4)$$

Clearly, $t_{ij} + m_{jr} \leq t_{ij} + M_{jr}$, because $m_{jr} \leq M_{jr}$. Moreover, $0 \leq t_{ij} + m_{jr}$, because neither of the terms is negative. So it is possible to find a t_{ir} that satisfies (4), therefore (2) can be satisfied.

Inequation (3) is equivalent to $m_{ir} \leq t_{ij} + t_{jr} \leq M_{ir}$ which is equivalent to

$$m_{ir} - t_{ij} \leq t_{jr} \leq M_{ir} - t_{ij} \quad (5)$$

Obviously, $m_{ir} - t_{ij} \leq M_{ir} - t_{ij}$, because $m_{ir} \leq M_{ir}$.

It is easy to show that $0 \leq M_{ir} - t_{ij}$:

\mathcal{D}_s^ξ is stable, so $M_{ij} + m_{jr} \leq M_{ir}$, therefore $m_{jr} \leq M_{ir} - M_{ij}$. Since $t_{ij} \leq M_{ij}$, $M_{ir} - M_{ij} \leq M_{ir} - t_{ij}$. But $0 \leq m_{jr}$, therefore $0 \leq M_{ir} - t_{ij}$.

So it is possible to find a t_{jr} that satisfies (5), therefore (3) can be satisfied.

Next, we must show that the same value of t_r can simultaneously satisfy both (2) and (3), or—equivalently—both (2) and (5). We do so by showing that $m_{jr} \leq M_{ir} - t_{ij}$ and $m_{ir} - t_{ij} \leq M_{jr}$.

\mathcal{D}_s^ξ is stable, so $M_{ij} + m_{jr} \leq M_{ir}$, therefore $m_{jr} \leq M_{ir} - M_{ij} \leq M_{ir} - t_{ij}$ (because $t_{ij} \leq M_{ij}$).

Similarly, $m_{ir} \leq m_{ij} + M_{jr}$, therefore $m_{ir} - m_{ij} \leq M_{jr}$. But $m_{ir} - t_{ij} \leq m_{ir} - m_{ij}$ (because $m_{ij} \leq t_{ij}$), so $m_{ir} - t_{ij} \leq M_{jr}$.

Intuitively, this means that none of the lower bounds on t_r imposed by (2) and (3) exceeds any of the upper bounds imposed by these inequations, for any choice of i and j that satisfies the assumptions. \square

For example, the sequence t_1t_2 , where $t_1 = 3$ and $t_2 = 4$, which is compatible with the stable table for scenario ξ of Fig. 4, can be extended to $t_0t_1t_2$, where $t_0 = 1$, or to $t_1t_2t_3$, where $t_3 = 6$, or to $t_0t_1t_2t_3$. All of the extended sequences are compatible with the stable table for ξ . The last one corresponds to a complete behaviour allowed by ξ : $(a, 1)(b, 3)(c, 4)(d, 6) \in \llbracket \xi \rrbracket$.

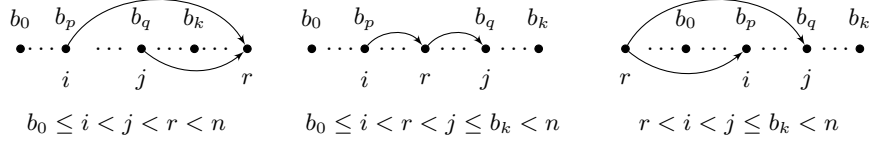


Fig. 5. Three cases of Theorem 1

Definition 2. Let \mathcal{D}_s^ξ be the stable distance table for scenario ξ of length n . Let $0 \leq i < j < n$, $m_{ij} \neq M_{ij}$ and $v \in I_{ij}^\xi$. The action of setting both l_{ij} and h_{ij} in \mathcal{D}_s^ξ to v is called collapsing I_{ij}^ξ to v . Then I_{ij}^ξ is a collapsed interval.

When an interval is collapsed to a value, the distance table may cease to be stable. Moreover, both the table and the interval are now associated with a different scenario, namely ξ augmented with the additional constraint $t_{ij} = v$.

Observation 2 Let \mathcal{D}_s^ξ be the stable distance table for scenario ξ of length n . After collapsing I_{ij}^ξ to a value in I_{ij}^ξ , \mathcal{D}_s^ξ can be successfully stabilised (i.e., stabilisation does not make the table invalid).

Proof. Let $v \in I_{ij}^\xi$. Collapsing I_{ij}^ξ to v is equivalent to imposing the additional constraints $t_{ij} \geq v, t_{ij} \leq v$. Consider the sequence $s = t_i t_j$, such that $t_i \geq m_{0i}^\xi$ and $t_j - t_i = v$. Clearly, s is compatible with \mathcal{D}_s^ξ . By Theorem 1, s can be extended to a complete behaviour $\mathcal{B} \in \llbracket \xi \rrbracket$. Behaviour \mathcal{B} corresponds to a scenario, say η , such that for $0 \leq r < s < n$, if $t_{rs}^\mathcal{B} = u$, then $\mathcal{D}^\eta[r, s] = (u, u)$. That is, $m_{rs}^\eta = M_{rs}^\eta = u$. We show that \mathcal{D}^η is stable:

For every $0 \leq i < j < k < n$: $m_{ij}^\eta = M_{ij}^\eta = t_{ij}^\mathcal{B}$, $m_{jk}^\eta = M_{jk}^\eta = t_{jk}^\mathcal{B}$, $m_{ik}^\eta = M_{ik}^\eta = t_{ik}^\mathcal{B}$. Inequation (1) becomes

$$t_{ij} + t_{jk} \leq t_{ik} \leq \left\{ \begin{array}{l} t_{ij} + t_{jk} \\ t_{ij} + t_{jk} \end{array} \right\} \leq t_{ik} \leq t_{ij} + t_{jk}$$

By definition, $t_{ij} + t_{jk} = (t_j - t_i) + (t_k - t_j) = t_k - t_i = t_{ik}$. So \mathcal{D}^η satisfies all the inequations. That is, \mathcal{D}^η is stable, which means that, after collapsing I_{ij}^ξ , stabilization could not have made \mathcal{D}_s^ξ invalid, since every interval in the stabilized table must contain the corresponding interval of \mathcal{D}^η . \square

It will sometimes be convenient to not annotate the symbols with the name of the scenario, i.e., to write I_{ij} and \mathcal{D} , instead of I_{ij}^ξ and \mathcal{D}^ξ .

Observation 3 Let \mathcal{D} be a stable distance table for a scenario of length n , $0 \leq i < j < n$, $0 \leq k < l < n$ and $i \neq k \vee j \neq l$. If I_{ij} and I_{kl} are collapsed intervals, then it is possible to construct a sequence $S = t_{b_0} t_{b_1} t_{b_2} t_{b_3}$ such that $\{b_0, b_1, b_2, b_3\} \subseteq \{i, j, k, l\}$ and S is compatible with \mathcal{D} .

Proof. We show how to construct a compatible sequence for each of three³ cases:

³ There are three other cases, which can be obtained from these by replacing i and j with k and l , and vice versa. The proofs can be obtained by interchanging i and k , as well as j and l .

1. $0 \leq k \leq i < j \leq l < n$
2. $0 \leq i < k < j < l < n$
3. $0 \leq i < j \leq k < l < n$

We show the proof for Case 1. The complete proof is presented elsewhere [24].

Case 1: $0 \leq k \leq i < j \leq l < n$ (see the diagrams in Fig. 6)

Let $k = b_0, i = b_1, j = b_2$ and $l = b_3$. Let $t_k \geq m_{0k}, t_i = t_k + m_{ki}, t_j = t_i + m_{ij}$ and $t_l = t_k + m_{kl}$.

We must show that (a) $t_{ki} \in I_{ki}$, (b) $t_{kj} \in I_{kj}$, (c) $t_{il} \in I_{il}$ and (d) $t_{jl} \in I_{jl}$ ⁴.

- (a) By definition, $t_{ki} = t_i - t_k = t_k + m_{ki} - t_k = m_{ki} \in I_{ki}$.
- (b) By definition, $t_{kj} = t_{ki} + t_{ij}$. Because I_{ij} is a collapsed interval, $t_{ij} = M_{ij}$. Therefore, $t_{kj} = m_{ki} + M_{ij}$. But $m_{kj} \leq m_{ki} + M_{ij} \leq M_{kj}$, so $t_{kj} \in I_{kj}$.
- (c) By definition, $t_{kl} = t_{ki} + t_{il}$. So $t_{il} = t_{kl} - t_{ki}$. Because I_{kl} is a collapsed interval, $t_{kl} = m_{kl}$. Therefore, $t_{il} = m_{kl} - m_{ki}$. We know $m_{ki} + m_{il} \leq m_{kl}$, so $m_{il} \leq m_{kl} - m_{ki}$. That is, $m_{il} \leq t_{il}$. We know $m_{kl} \leq m_{ki} + M_{il}$, so $m_{kl} - m_{ki} \leq M_{il}$. Therefore, $t_{il} \leq M_{il}$.
- (d) By definition, $t_{il} = t_{ij} + t_{jl}$, so $t_{jl} = t_{il} - t_{ij} = m_{kl} - m_{ki} - t_{ij}$. I_{ij} is a collapsed interval: $t_{ij} = m_{ij}$. So $t_{jl} = m_{kl} - (m_{ki} + m_{ij})$. We know $m_{ki} + m_{ij} \leq m_{kj}$, so, $t_{jl} \geq m_{kl} - m_{kj}$. But $m_{kj} + m_{jl} \leq m_{kl}$, so $m_{kl} - m_{kj} \geq m_{jl}$. Therefore, $t_{jl} \geq m_{jl}$. I_{ij} is a collapsed interval: $t_{ij} = M_{ij}$. So $t_{jl} = m_{kl} - (m_{ki} + M_{ij})$. We know $m_{kj} \leq m_{ki} + M_{ij}$, so, $t_{jl} \leq m_{kl} - m_{kj}$. But $m_{kl} \leq m_{kj} + M_{jl}$, so $m_{kl} - m_{kj} \leq M_{jl}$. Therefore, $t_{jl} \leq M_{jl}$. \square

4 z_pairs

In our earlier work [13] we proved that if $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$ are two optimized scenarios of length n , such that $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then there are constraints $\alpha = \tau_{i,j} \sim a \in \mathcal{C}_1$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}_2$ (for some $a, b \in \mathbb{Q}$, $0 \leq i < j < n$, $0 \leq k < l < n$), such that $\alpha \notin \mathcal{C}_2$, $\beta \notin \mathcal{C}_1$ and $(i \neq k \text{ or } j \neq l)$. That is, the absence of such a pair of explicit constraints implies that there is no zigzagging.

It turns out that in the presence of zigzagging behaviours the constraints of ξ and η are involved in some interesting relations, which depend on the positions of i, j, k and l with respect to each other. This will allow us to formulate a stronger condition: zigzagging is present if and only if the condition is satisfied (Theorem 2 and Theorem 3).

Definition 3. Let ξ and η be two scenarios of length n . If behaviour $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$ is such that $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^\xi, t_{ij}^{\mathcal{B}^z} \in I_{ij}^\eta, t_{kl}^{\mathcal{B}^z} \in I_{kl}^\xi, t_{kl}^{\mathcal{B}^z} \notin I_{kl}^\eta$ and $i \neq k \vee j \neq l$ ($0 \leq i < j < n, 0 \leq k < l < n$), then we say \mathcal{B}^z zigzags through ij and kl .

As an example consider ξ of Fig. 1 and η of Fig. 2. Behaviour $\mathcal{B}^z = (a, 0)(b, 2)(c, 5)$ zigzags through 01 and 12: $t_{01}^{\mathcal{B}^z} \notin I_{01}^\xi, t_{01}^{\mathcal{B}^z} \in I_{01}^\eta, t_{12}^{\mathcal{B}^z} \notin I_{12}^\eta$ and $t_{12}^{\mathcal{B}^z} \in I_{12}^\xi$.

⁴ If $k = i$, then $t_{ki} = 0$, hence $t_{ki} \in I_{ki} = [0, 0]$. Similarly, if $j = l$, then $t_{jl} = 0$, hence $t_{jl} \in I_{jl} = [0, 0]$.



Fig. 6. case 1 of Observation 3 and of Definition 4

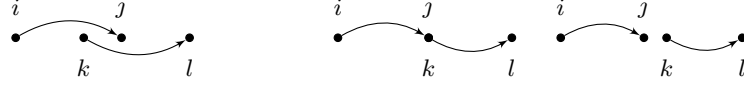


Fig. 7. cases 2, 3 of Observation 3 and of Definition 4

Definition 4. Let ξ and η be two scenarios of length n with the same sequences of events, such that $\xi \not\subseteq \eta$, $\eta \not\subseteq \xi$ and $\xi \uplus \eta$ is defined. Let $\alpha = \tau_{i,j} \sim a \in \mathcal{C}(\mathcal{D}_s^\xi)$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}(\mathcal{D}_s^\eta)$ ($a, b \in \mathbb{Q}$), such that $i \neq k \vee j \neq l$, $\alpha \notin \mathcal{C}(\mathcal{D}_s^\eta)$ and $\beta \notin \mathcal{C}(\mathcal{D}_s^\xi)$. Constraints α and β form a z_pair if one of the following conditions holds⁵:

- (1) $0 \leq k \leq i < j \leq l < n$ and
 - (a) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \geq b$, $m_{ij}^\eta < a$, $m_{kl}^\xi < b$, or
 - (b) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \leq b$, $m_{ij}^\eta < a$, $b < M_{kl}^\xi$, and additionally $M_{ki}^{\xi \uplus \eta} + a + M_{jl}^{\xi \uplus \eta} > b$, or
 - (c) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \geq b$, $a < M_{ij}^\eta$, $m_{kl}^\xi < b$, and additionally $m_{ki}^{\xi \uplus \eta} + a + m_{jl}^{\xi \uplus \eta} < b$, or
 - (d) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \leq b$, $a < M_{ij}^\eta$, $b < M_{kl}^\xi$.
- (2) $0 \leq i < k < j < l < n$ and
 - (a) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \geq b$, $m_{ij}^\eta < a$, $m_{kl}^\xi < b$, and additionally $m_{il}^{\xi \uplus \eta} - a < b - m_{kj}^{\xi \uplus \eta}$, or
 - (b) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \leq b$, $m_{ij}^\eta < a$, $b < M_{kl}^\xi$, and additionally $a + M_{jl}^{\xi \uplus \eta} > m_{ik}^{\xi \uplus \eta} + b$, or
 - (c) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \geq b$, $a < M_{ij}^\eta$, $m_{kl}^\xi < b$, and additionally $a + m_{jl}^{\xi \uplus \eta} < M_{ik}^{\xi \uplus \eta} + b$, or
 - (d) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \leq b$, $a < M_{ij}^\eta$, $b < M_{kl}^\xi$, and additionally $M_{il}^{\xi \uplus \eta} - a > b - M_{kj}^{\xi \uplus \eta}$.
- (3) $0 \leq i < j \leq k < l < n$ and
 - (a) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \geq b$, $m_{ij}^\eta < a$, $m_{kl}^\xi < b$, or
 - (b) $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \leq b$, $m_{ij}^\eta < a$, $b < M_{kl}^\xi$, or
 - (c) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \geq b$, $a < M_{ij}^\eta$, $m_{kl}^\xi < b$, or
 - (d) $\alpha = \tau_{i,j} \leq a$, $\beta = \tau_{k,l} \leq b$, $a < M_{ij}^\eta$, $b < M_{kl}^\xi$.

⁵ See the diagrams in Fig. 6 and Fig. 7.

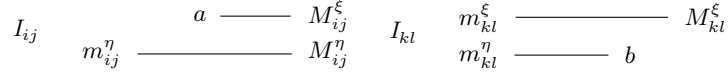


Fig. 8. Definition 4 cases (1)(b), (2)(b) and (3)(b): $\tau_{i,j} \geq a$, $\tau_{k,l} \leq b$, $m_{ij}^\eta < a$, $b < M_{kl}^\xi$

In each case there are four other subcases which can be obtained by interchanging ξ and η .

The three cases of Definition 4 cover all⁶ the possibilities of the positions of i , j , k and l with respect to each other. The “main” subcases of each case cover all the possible combinations of the forms of α and β , and the relations between the minima and maxima of distances between events i and j and between events k and l in ξ and η . Intuitively, the conditions capture all the possibilities for $I_{ij}^\eta \setminus I_{ij}^\xi \neq \emptyset$ and $I_{kl}^\xi \setminus I_{kl}^\eta \neq \emptyset$, to guarantee “there is room” for behaviours to zigzag through ij and kl . But these are only half of the possibilities: The cases for $I_{ij}^\xi \setminus I_{ij}^\eta \neq \emptyset$ and $I_{kl}^\eta \setminus I_{kl}^\xi \neq \emptyset$ are obtained by exchanging ξ and η .

Fig. 8 is a schematic illustration of the main conditions of cases (1)(b), (2)(b) and (3)(b): $\alpha = \tau_{i,j} \geq a$, $\beta = \tau_{k,l} \leq b$, $m_{ij}^\eta < a$, $b < M_{kl}^\xi$. Because of constraint α , $m_{ij}^\xi = a$ and because of constraint β , $M_{kl}^\eta = b$. Therefore, $I_{ij}^\eta \setminus I_{ij}^\xi = \{u \in \mathbb{Q}_{\geq 0} \mid m_{kl}^\eta \leq u < a\} \neq \emptyset$ and $I_{kl}^\xi \setminus I_{kl}^\eta = \{v \in \mathbb{Q}_{\geq 0} \mid b < v < M_{kl}^\xi\} \neq \emptyset$.

The *additional* conditions specify certain relations that must hold between various minima and maxima in ξ and η for there to be zigzagging behaviours.

It is worth mentioning that if α and β form a z_pair , then neither of them can be a default constraint.

In the rest of the paper we are going to prove theorems that will show the following relations between the existence of a z_pair and the existence of the union of two scenarios:

- no $z_pair \Rightarrow$ no zigzagging \Rightarrow union (Theorem 2);
- $z_pair \Rightarrow$ zigzagging \Rightarrow no union (Theorem 3).

5 A sufficient condition for the existence of union

Theorem 2. *Let ξ and η be two scenarios of length n with the same sequences of events, such that $\xi \not\subseteq \eta$ and $\eta \not\subseteq \xi$. If $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$ zigzags through ij and kl , then there exist $\alpha = \tau_{i,j} \sim a \in \mathcal{C}(\mathcal{D}_s^\xi)$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}(\mathcal{D}_s^\eta)$ ($a, b \in \mathbb{Q}$) such that α and β form a z_pair .*

⁶ In the definition it is assumed that ij is within ξ and kl is within η . There are three other cases where ij is within η and kl is within ξ . Those cases are symmetric and can be obtained by interchanging ξ and η .

Proof. We show the proof for case (1) of Definition 4: $0 \leq k \leq i < j \leq l < n$. The complete proof is presented elsewhere [24].

Let \mathcal{B}^z be a behaviour in $\mathcal{Z}(\xi, \eta)$ that zigzags through ij and kl . There are two cases to consider:

- (1) $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^\xi$, $t_{ij}^{\mathcal{B}^z} \in I_{ij}^\eta$, $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^\eta$ and $t_{kl}^{\mathcal{B}^z} \in I_{kl}^\xi$.

Since $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^\xi$, one of the following two cases must hold:

- (a) $t_{ij}^{\mathcal{B}^z} < m_{ij}^\xi$. Since $t_{ij}^{\mathcal{B}^z} \in I_{ij}^\eta$, we must have $m_{ij}^\eta \leq t_{ij}^{\mathcal{B}^z}$. It follows that $m_{ij}^\eta < m_{ij}^\xi$. Therefore $m_{ij}^\xi > 0$: there exists a non-default constraint $\alpha = \tau_{i,j} \geq a$ in $\mathcal{C}(\mathcal{D}_s^\xi)$, where $a = m_{ij}^\xi$.

Since $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^\eta$, one of the following two cases must hold:

- $t_{kl}^{\mathcal{B}^z} < m_{kl}^\eta$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^\xi$, we must have $m_{kl}^\xi \leq t_{kl}^{\mathcal{B}^z}$. It follows that $m_{kl}^\xi < m_{kl}^\eta$. Therefore $m_{kl}^\eta > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}_s^\eta)$, where $b = m_{kl}^\eta$.

So α and β form a z -pair (case (1)(a) of Definition 4).

- $M_{kl}^\eta < t_{kl}^{\mathcal{B}^z}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^\xi$, we must have $t_{kl}^{\mathcal{B}^z} \leq M_{kl}^\xi$. It follows that $M_{kl}^\eta < M_{kl}^\xi$. Therefore $M_{kl}^\eta < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}_s^\eta)$, where $b = M_{kl}^\eta$.

The constraints α and β fit case (1)(b) of Definition 4. We must show $M_{ki}^{\xi \uplus \eta} + a + M_{jl}^{\xi \uplus \eta} > b$.

Assume $M_{ki}^{\xi \uplus \eta} + a + M_{jl}^{\xi \uplus \eta} \leq b$.

By definition, $t_{kl}^{\mathcal{B}^z} = t_{ki}^{\mathcal{B}^z} + t_{ij}^{\mathcal{B}^z} + t_{jl}^{\mathcal{B}^z}$. Because $\mathcal{B}^z \in \llbracket \xi \uplus \eta \rrbracket$, we have $t_{ki}^{\mathcal{B}^z} \leq M_{ki}^{\xi \uplus \eta}$ and $t_{jl}^{\mathcal{B}^z} \leq M_{jl}^{\xi \uplus \eta}$. Therefore, $t_{kl}^{\mathcal{B}^z} \leq M_{ki}^{\xi \uplus \eta} + t_{ij}^{\mathcal{B}^z} + M_{jl}^{\xi \uplus \eta}$.

Since $t_{ij}^{\mathcal{B}^z} < m_{ij}^\xi$ and $m_{ij}^\xi = a$, we have $t_{kl}^{\mathcal{B}^z} < M_{ki}^{\xi \uplus \eta} + a + M_{jl}^{\xi \uplus \eta}$. By the assumption, $M_{ki}^{\xi \uplus \eta} + a + M_{jl}^{\xi \uplus \eta} \leq b$. So it follows that $t_{kl}^{\mathcal{B}^z} < b$.

But $M_{kl}^\eta < t_{kl}^{\mathcal{B}^z}$, so $M_{kl}^\eta < b$: a contradiction.

So α and β form a z -pair (case (1)(b) of Definition 4).

- (b) $M_{ij}^\xi < t_{ij}^{\mathcal{B}^z}$. Since $t_{ij}^{\mathcal{B}^z} \in I_{ij}^\eta$, we must have $t_{ij}^{\mathcal{B}^z} \leq M_{ij}^\eta$. It follows that $M_{ij}^\xi < M_{ij}^\eta$. Therefore $M_{ij}^\xi < \infty$: there exists a non-default constraint $\alpha = \tau_{i,j} \leq a$ in $\mathcal{C}(\mathcal{D}_s^\xi)$, where $a = M_{ij}^\xi$.

Since $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^\eta$, one of the following two cases must hold:

- $t_{kl}^{\mathcal{B}^z} < m_{kl}^\eta$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^\xi$, we must have $m_{kl}^\xi \leq t_{kl}^{\mathcal{B}^z}$. It follows that $m_{kl}^\xi < m_{kl}^\eta$. Therefore $m_{kl}^\eta > 0$: there exists a non-default constraint $\beta = \tau_{k,l} \geq b \in \mathcal{C}(\mathcal{D}_s^\eta)$, where $b = m_{kl}^\eta$.

The constraints α and β fit case (1)(c) of Definition 4. We must show $m_{ki}^{\xi \uplus \eta} + a + m_{jl}^{\xi \uplus \eta} < b$.

Assume $m_{ki}^{\xi \uplus \eta} + a + m_{jl}^{\xi \uplus \eta} \geq b$.

By definition, $t_{kl}^{\mathcal{B}^z} = t_{ki}^{\mathcal{B}^z} + t_{ij}^{\mathcal{B}^z} + t_{jl}^{\mathcal{B}^z}$. Because $\mathcal{B}^z \in \llbracket \xi \uplus \eta \rrbracket$, we have $t_{ki}^{\mathcal{B}^z} \geq m_{ki}^{\xi \uplus \eta}$ and $t_{jl}^{\mathcal{B}^z} \geq m_{jl}^{\xi \uplus \eta}$. Therefore, $t_{kl}^{\mathcal{B}^z} \geq m_{ki}^{\xi \uplus \eta} + t_{ij}^{\mathcal{B}^z} + m_{jl}^{\xi \uplus \eta}$.

Since $t_{ij}^{\mathcal{B}^z} > M_{ij}^\xi$ and $M_{ij}^\xi = a$, we have $t_{kl}^{\mathcal{B}^z} > m_{ki}^{\xi \uplus \eta} + a + m_{jl}^{\xi \uplus \eta}$. By

- the assumption, $m_{ki}^{\xi \uplus \eta} + a + m_{jl}^{\xi \uplus \eta} \geq b$. So it follows that $t_{kl}^{\mathcal{B}^z} > b$.
 But $m_{kl}^\eta > t_{kl}^{\mathcal{B}^z}$, so $m_{kl}^\eta > b$: a contradiction.
 So α and β form a z_pair (case (1)(c) of Definition 4).
 – $M_{kl}^\eta < t_{kl}^{\mathcal{B}^z}$. Since $t_{kl}^{\mathcal{B}^z} \in I_{kl}^\xi$, we must have $t_{kl}^{\mathcal{B}^z} \leq M_{kl}^\xi$. It follows that $M_{kl}^\eta < M_{kl}^\xi$. Therefore $M_{kl}^\eta < \infty$: there exists a non-default constraint $\beta = \tau_{k,l} \leq b \in \mathcal{C}(\mathcal{D}_s^\eta)$, where $b = M_{kl}^\eta$.
 So α and β form a z_pair (case (1)(d) of Definition 4).
 (2) $t_{ij}^{\mathcal{B}^z} \in I_{ij}^\xi$, $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^\eta$, $t_{kl}^{\mathcal{B}^z} \in I_{kl}^\eta$ and $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^\xi$.
 The proof can be obtained by exchanging ξ and η in case (1). □

As an example consider ξ and η of figures 1 and 2 along with their stable distance tables once more. As we mentioned before, $\mathcal{Z}(\xi, \eta) \neq \emptyset$ and there is a behaviour that zigzags through 01 and 12. For example, $\mathcal{B}^z = (a, 0)(b, 1)(c, 4)$ is such a behaviour: $t_{01}^{\mathcal{B}^z} = 1 - 0 \notin I_{01}^\xi$ and $t_{12}^{\mathcal{B}^z} = 4 - 1 \notin I_{12}^\eta$. According to Theorem 2 at least one z_pair must exist between ξ and η . Indeed, constraints $\alpha = \tau_{0,1} \geq 3$ in ξ and $\beta = \tau_{1,2} \geq 4$ in η satisfy the requirements of Definition 4 (case 3.a), and therefore, form a z_pair : $i = 0, j = k = 1, l = 2, i < j = k < l, a = 3, b = 4, m_{01}^\eta = 0 < a = 3$ and $m_{12}^\xi = 0 < b = 4$.

As another example consider scenarios ξ and η of Fig. 4. Constraints $\tau_{0,2} \leq 3$ in ξ and $\tau_{0,3} \geq 2$ in η do not form a z_pair : $i = k = 0, j = 2, l = 3$, and $a = 3 < M_{02}^\eta = \infty, m_{03}^\xi = 0 < b = 2$ (case 1.c), but $m_{02}^{\xi \uplus \eta} + a + m_{23}^{\xi \uplus \eta} = 0 + 3 + 0 \not\leq b = 2$. Constraints $\tau_{0,3} \leq 5$ in ξ and $\tau_{1,3} \leq 5$ in η do not form a z_pair : $I_{13}^\xi = I_{13}^\eta$. Constraints $\tau_{0,2} \leq 3$ in ξ and $\tau_{1,3} \leq 5$ in η do not form a z_pair for the same reason. Of course, constraints $\tau_{0,3} \leq 5$ in ξ and $\tau_{0,3} \geq 2$ in η do not form a z_pair : $i = k = 0, j = l = 3$, so $i = k \wedge j = l$.

In fact, none of the constraints of ξ and η satisfy the conditions of any of the cases of Definition 4: there is no z_pair between constraints of ξ and η . This is in accordance with Theorem 2: as we mentioned before, $\mathcal{Z}(\xi, \eta) = \emptyset$.

The consequence of Theorem 2 is that if there is no z_pair between scenarios ξ and η , then $\mathcal{Z}(\xi, \eta) = \emptyset$, therefore $\xi \cup \eta$ exists. That is, the non-existence of z_pairs between two scenarios is a sufficient condition for the existence of the union between the two (provided $\xi \uplus \eta$ is defined).

In our earlier work [13] we identified another such condition: if scenarios ξ and η are optimized and $\mathcal{Z}(\xi, \eta) \neq \emptyset$, then ξ and η must each have at least one explicit constraint that the other does not have, such that the two constraints are not between the same events in both ξ and η . The absence of such a pair of explicit constraints is also a sufficient condition for the existence of $\xi \cup \eta$, but the pair need not be a z_pair .

The existence of a z_pair turns out to be not only a necessary, but also a sufficient condition for the existence of zigzagging behaviours (see Theorem 3).

Consider scenarios ξ of Fig. 9 and η of Fig. 10: $\mathcal{Z}(\xi, \eta) \neq \emptyset$. Indeed, constraints $\tau_{0,2} \leq 2$ of ξ and $\tau_{1,3} \geq 4$ of η are two different explicit constraints between different events, events 0 and 2, and events 1 and 3, respectively. However, the two constraints do not form a z_pair : their form fits case (2)(c) of Definition 4,

0 : a ;
1 : $b \{\tau_{0,1} \geq 1\}$;
2 : $c \{\tau_{0,2} \leq 2\}$;
3 : d .

	1	2	3
0	(1, 2)	(1, 2)	(1, ∞)
1		(0, 1)	(0, ∞)
2			(0, ∞)

Fig. 9. ξ and its stable table

0 : a ;
1 : $b \{\tau_{0,1} \leq 1\}$;
2 : $c \{\tau_{1,2} \leq 1\}$;
3 : $d \{\tau_{1,3} \geq 4\}$.

	1	2	3
0	(0, 1)	(0, 2)	(4, ∞)
1		(0, 1)	(4, ∞)
2			(3, ∞)

Fig. 10. η and its stable table

however $a = 2 \not\leq M_{02}^\eta = 2$. But, of course, there must exist at least one z_pair . Constraint $\tau_{0,2} \geq 1$ of ξ , which is an implied constraint, and constraint $\tau_{1,3} \geq 4$ of η form a z_pair .

6 A sufficient condition for the non-existence of union

Our goal is to show that the existence of a z_pair between two scenarios ξ and η is a sufficient condition for the non-existence of $\xi \cup \eta$, because of the existence of zigzagging behaviours between the two scenarios. More precisely, if there is a z_pair between ξ and η , then there exist $0 \leq i < j < n$, $0 \leq k < l < n$, $u \in I_{ij}^{\xi \cup \eta}$, $v \in I_{kl}^{\xi \cup \eta}$ and behaviour $\mathcal{B}^z \in \llbracket \xi \cup \eta \rrbracket$, such that $t_{ij}^{\mathcal{B}^z} = u \in I_{ij}^\xi \setminus I_{ij}^\eta$ and $t_{kl}^{\mathcal{B}^z} = v \in I_{kl}^\eta \setminus I_{kl}^\xi$.

Observation 4 Let \mathcal{D} be a stable distance table for a scenario of length n and let $0 \leq i \leq j \leq k < n$ be integers.

Let t_{ij} satisfy $m_{ij} \leq t_{ij} \leq M_{ij}$. Then, after replacing m_{ij} and M_{ij} with t_{ij} and stabilising the table, if m_{ik} changes, it will increase to at most $t_{ij} + m_{jk}$.

Proof. The proof is presented in the longer version of the paper [24].

Observation 5 Let \mathcal{D} be a stable distance table for a scenario of length n and let $0 \leq i \leq j \leq k < n$ be integers. Let t_{ij} satisfy $m_{ij} \leq t_{ij} \leq M_{ij}$. Then, after replacing m_{ij} and M_{ij} with t_{ij} , if the inequation $t_{ij} + m_{jk} \leq m_{ik}$ holds, then during stabilisation m_{ik} will not change.

Proof. Assume m_{ik} will change during stabilisation. In that case, $m_{ik} < \mathbf{m}_{ik}$, and By Observation 4, $\mathbf{m}_{ik} \leq t_{ij} + m_{jk}$.

By the assumption, the inequation $t_{ij} + m_{jk} \leq m_{ik}$ holds. It follows that $\mathbf{m}_{ik} \leq m_{ik}$: a contradiction with $m_{ik} < \mathbf{m}_{ik}$. \square

Theorem 3. Let ξ and η be two scenarios of length n with the same sequences of events, such that $\xi \not\subseteq \eta$, $\eta \not\subseteq \xi$ and $\xi \cup \eta$ is defined. If there are $\alpha = \tau_{i,j} \sim a \in \mathcal{C}(\mathcal{D}_s^\xi)$ and $\beta = \tau_{k,l} \sim b \in \mathcal{C}(\mathcal{D}_s^\eta)$ ($a, b \in \mathbb{Q}$), such that α and β form a z_pair , then there is a $\mathcal{B}^z \in \mathcal{Z}(\xi, \eta)$, such that \mathcal{B}^z zigzags through ij and kl .

Proof. Just as in Definition 4, there are three cases, determined by the relative positions of i, j, k and l . Each of these cases has four subcases, determined by the relations between the minima and maxima of the relevant intervals.

We show the proof for case (1)(a). The complete proof is presented elsewhere [24].

Case (1)(a): $0 \leq k \leq i < j \leq l < n$ ($i \neq k \vee j \neq l$)

$$\alpha = \tau_{i,j} \geq a, \beta = \tau_{k,l} \geq b, m_{ij}^\eta < a, m_{kl}^\xi < b$$

We show the proof for the case $0 \leq k < i < j < l < n$. The proofs for $k = i$ or $j = l$ are special cases of the proof shown below.

We must show that there is a behaviour $\mathcal{B}^z \in \llbracket \xi \uplus \eta \rrbracket$ such that \mathcal{B}^z zigzags through ij and kl . Because ij is entirely within kl , i.e., $k < i < j < l$, we must have $t_{ij}^{\mathcal{B}^z} \leq t_{kl}^{\mathcal{B}^z}$.

Because of constraint α , $m_{ij}^\xi = a$ and because of constraint β , $m_{kl}^\eta = b$. $m_{ij}^{\xi \uplus \eta} = \min(m_{ij}^\xi, m_{ij}^\eta) = m_{ij}^\eta$ and $m_{kl}^{\xi \uplus \eta} = \min(m_{kl}^\xi, m_{kl}^\eta) = m_{kl}^\xi$.

Both $I_{ij}^{\xi \uplus \eta} \setminus I_{ij}^\xi = \{u \in \mathbb{Q}_{>0} \mid m_{ij}^\eta \leq u < a\}$ and $I_{kl}^{\xi \uplus \eta} \setminus I_{kl}^\eta = \{w \in \mathbb{Q}_{\geq 0} \mid m_{kl}^\xi \leq w < b\}$ are non-empty. So it is possible to pick a value $t_{ij} = a - \delta$ ($\delta \in \mathbb{Q}_{>0}$) in the interval $I_{ij}^{\xi \uplus \eta} \setminus I_{ij}^\xi$.

Next we must show that after collapsing $I_{ij}^{\xi \uplus \eta}$ of $\mathcal{D}_s^{\xi \uplus \eta}$ to t_{ij} (i.e., setting $m_{ij}^{\xi \uplus \eta} = M_{ij}^{\xi \uplus \eta} = t_{ij}$ and stabilizing $\mathcal{D}_s^{\xi \uplus \eta}$), hence obtaining $\mathcal{D}_s^{\xi \uplus \eta}$, it is still possible to pick a value t_{kl} in the interval $I_{kl}^{\xi \uplus \eta}$ of table $\mathcal{D}_s^{\xi \uplus \eta}$, such that t_{kl} belongs to I_{kl}^ξ , but does not belong to I_{kl}^η , and that $t_{ij} \leq t_{kl}$.

(We use bold font to distinguish items that are updated during stabilization.)

In $\mathcal{D}_s^{\xi \uplus \eta}$ we have $m_{ki}^{\xi \uplus \eta} + m_{ij}^{\xi \uplus \eta} \leq m_{kj}^{\xi \uplus \eta}$ and $m_{kj}^{\xi \uplus \eta} + m_{jl}^{\xi \uplus \eta} \leq m_{kl}^{\xi \uplus \eta}$.

We set $\mathbf{m}_{ij}^{\xi \uplus \eta} = M_{ij}^{\xi \uplus \eta} = t_{ij}$ in interval $I_{ij}^{\xi \uplus \eta}$ of $\mathcal{D}_s^{\xi \uplus \eta}$ stabilise it to obtain $\mathcal{D}_s^{\xi \uplus \eta}$. After this one of the following two things will happen:

- (i) The inequation $m_{ki}^{\xi \uplus \eta} + t_{ij} \leq m_{kj}^{\xi \uplus \eta}$ holds, in which case, by Observation 5, $m_{kj}^{\xi \uplus \eta}$ will not change: $\mathbf{m}_{kj}^{\xi \uplus \eta} = m_{kj}^{\xi \uplus \eta}$. The inequation $m_{kj}^{\xi \uplus \eta} + m_{jl}^{\xi \uplus \eta} \leq m_{kl}^{\xi \uplus \eta}$ will then continue to hold. Therefore $m_{kl}^{\xi \uplus \eta}$ will not change: $\mathbf{m}_{kl}^{\xi \uplus \eta} = m_{kl}^{\xi \uplus \eta} = m_{kl}^\xi$. Hence, $\mathbf{I}_{kl}^{\xi \uplus \eta} \setminus I_{kl}^\eta = \{w \in \mathbb{Q}_{\geq 0} \mid m_{kl}^\xi \leq w < b\}$.
- (ii) The inequation $m_{ki}^{\xi \uplus \eta} + t_{ij} \leq m_{kj}^{\xi \uplus \eta}$ does not hold. In that case, to restore the inequation, $m_{kj}^{\xi \uplus \eta}$ will be set to $m_{ki}^{\xi \uplus \eta} + t_{ij}$. By Observation 4, $m_{kj}^{\xi \uplus \eta}$ cannot further increase, therefore $\mathbf{m}_{kj}^{\xi \uplus \eta} = m_{ki}^{\xi \uplus \eta} + a - \delta$. But since $\mathbf{m}_{kj}^{\xi \uplus \eta}$ has changed, the inequation $\mathbf{m}_{kj}^{\xi \uplus \eta} + m_{jl}^{\xi \uplus \eta} \leq m_{kl}^{\xi \uplus \eta}$ might not hold. There are two cases to consider:
 1. If the new value of $\mathbf{m}_{kj}^{\xi \uplus \eta}$ does not affect the satisfiability of the inequation $\mathbf{m}_{kj}^{\xi \uplus \eta} + m_{jl}^{\xi \uplus \eta} \leq m_{kl}^{\xi \uplus \eta}$, then $m_{kl}^{\xi \uplus \eta}$ will remain the same, so just as in case (i), $\mathbf{I}_{kl}^{\xi \uplus \eta} \setminus I_{kl}^\eta = \{w \in \mathbb{Q}_{\geq 0} \mid m_{kl}^\xi \leq w < b\}$.
 2. If $\mathbf{m}_{kj}^{\xi \uplus \eta} + m_{jl}^{\xi \uplus \eta} \leq m_{kl}^{\xi \uplus \eta}$ is not satisfied, the inequation will be restored by setting $m_{kl}^{\xi \uplus \eta}$ to $\mathbf{m}_{kj}^{\xi \uplus \eta} + m_{jl}^{\xi \uplus \eta}$: $\mathbf{m}_{kl}^{\xi \uplus \eta} := \mathbf{m}_{kj}^{\xi \uplus \eta} + m_{jl}^{\xi \uplus \eta} = m_{ki}^{\xi \uplus \eta} + a - \delta + m_{jl}^{\xi \uplus \eta}$. Now the inequation $\mathbf{m}_{kj}^{\xi \uplus \eta} + m_{jl}^{\xi \uplus \eta} \leq \mathbf{m}_{kl}^{\xi \uplus \eta}$ holds, and by Observation 5, $\mathbf{m}_{kl}^{\xi \uplus \eta}$ cannot be further increased. Next, we show that $\mathbf{m}_{kl}^{\xi \uplus \eta} < b$. For that, we show that $m_{ki}^{\xi \uplus \eta} + a + m_{jl}^{\xi \uplus \eta} < b$.

Assume $b \leq m_{ki}^{\xi \cup \eta} + a + m_{jl}^{\xi \cup \eta}$. Because $m_{kl}^{\xi} < b$, we will have $m_{kl}^{\xi} < m_{ki}^{\xi \cup \eta} + a + m_{jl}^{\xi \cup \eta}$. Since $m_{ki}^{\xi \cup \eta} \leq m_{ki}^{\xi}$ and $m_{jl}^{\xi \cup \eta} \leq m_{jl}^{\xi}$, we will have $m_{kl}^{\xi} < m_{ki}^{\xi} + a + m_{jl}^{\xi}$.

But $m_{ki}^{\xi} + m_{il}^{\xi} \leq m_{kl}^{\xi}$, so $m_{ki}^{\xi} + m_{il}^{\xi} < m_{ki}^{\xi} + a + m_{jl}^{\xi}$. Therefore $m_{il}^{\xi} < a + m_{jl}^{\xi}$. But $m_{ij}^{\xi} + m_{jl}^{\xi} \leq m_{il}^{\xi}$, and $m_{ij}^{\xi} = a$. So $a + m_{jl}^{\xi} < a + m_{jl}^{\xi}$: a contradiction.

Because $m_{ki}^{\xi \cup \eta} + a + m_{jl}^{\xi \cup \eta} < b$, we will have $m_{kl}^{\xi \cup \eta} < b$.

So $\mathbf{I}_{kl}^{\xi \cup \eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid \mathbf{m}_{kl}^{\xi \cup \eta} \leq w < b\}$ will be non-empty.

If (i) or (ii).1, because ξ is consistent, we have $m_{ij}^{\xi} \leq m_{kl}^{\xi}$. Since $t_{ij} = a - \delta$, any value that is picked for t_{kl} from $\mathbf{I}_{kl}^{\xi \cup \eta} \setminus I_{kl}^{\eta}$ will satisfy $t_{ij} \leq t_{kl}$.

If (ii).2, since $\mathbf{m}_{kl}^{\xi \cup \eta} = m_{ki}^{\xi \cup \eta} + t_{ij} + m_{jl}^{\xi \cup \eta}$, we have $t_{ij} \leq \mathbf{m}_{kl}^{\xi \cup \eta}$. Then any value t_{kl} in $\mathbf{I}_{kl}^{\xi \cup \eta} \setminus I_{kl}^{\eta} = \{w \in \mathbb{Q}_{\geq 0} \mid \mathbf{m}_{kl}^{\xi \cup \eta} \leq w < b\}$ will satisfy $t_{ij} \leq t_{kl}$.

Obviously, $t_{ij} \in \mathbf{I}_{ij}^{\xi \cup \eta}$ and $t_{kl} \in \mathbf{I}_{kl}^{\xi \cup \eta}$. Moreover, $0 \leq k < i < j < l < n$, so, by Observation 3, it is possible to form a sequence $t_k \leq t_i \leq t_j \leq t_l$ that is compatible with $\mathbf{D}_s^{\xi \cup \eta}$. By Theorem 1, S can be extended to a behaviour, say \mathcal{B}^z , in $\llbracket \xi \cup \eta \rrbracket$. Clearly, $t_{ij}^{\mathcal{B}^z} \notin I_{ij}^{\xi}$ and $t_{kl}^{\mathcal{B}^z} \notin I_{kl}^{\eta}$. Therefore, \mathcal{B}^z zigzags through ij and kl . \square

The consequence of Theorem 3 is that if ξ and η have constraints that form a z -pair, then $\xi \cup \eta$ does not exist.

7 Conclusions

In our earlier work [13] we had found a sufficient condition for the existence of the union of two optimized timed scenarios $\xi = (\mathcal{E}, \mathcal{C}_1)$ and $\eta = (\mathcal{E}, \mathcal{C}_2)$: if \mathcal{C}_1 and \mathcal{C}_2 do not contain a pair of constraints $\alpha = \tau_{i,j} \sim a$ and $\beta = \tau_{k,l} \sim b$, respectively, such that $i \neq k$ or $j \neq l$, $\alpha \notin \mathcal{C}_2$ and $\beta \notin \mathcal{C}_1$, then there will be no “zigzagging” behaviours between ξ and η , and therefore $\xi \cup \eta$ exists. Zigzagging behaviours belong neither to $\llbracket \xi \rrbracket$, nor to $\llbracket \eta \rrbracket$, even though they belong to $\llbracket \xi \cup \eta \rrbracket$, i.e., the semantics of the combination (or the “quasi-union”) of ξ and η .

In the current paper we investigate, in more depth, the conditions under which the union of the sets of behaviours allowed by two consistent scenarios, ξ and η , can be represented by a single scenario, namely the union $\xi \cup \eta$. Our investigation reveals that in the presence of zigzagging behaviours the constraints of ξ and η must satisfy certain additional criteria. Based on this observation we formulate a sufficient and necessary condition for the existence of the union (Theorem 2 and Theorem 3).

The union operation is directly relevant to the problem of synthesizing timed automata with minimal numbers of clocks from a set of scenarios [14]. It is also relevant to various model-checking techniques based on timed automata, if stable distance tables are used to capture zones corresponding to locations in the automata.

A detailed comparison of timed scenarios with other related work, in particular with Difference Bounds Matrices (DBMs) [25], can be found in our earlier work [10, 12]. A union operation has been defined for DBMs, which are used for representing zones in timed automata. However, the union of two zones (represented by DBMs) is—in general—a non-convex set, and therefore cannot be represented by a DBM. So the union of two zones has to be approximated, e.g., by convex hulls [18], or some other safe abstraction [17]. Another method for checking whether the union of two DBMs is itself a DBM has been developed using convex hulls together with Clock Difference Diagrams (CDDs) [26].

Our necessary and sufficient condition provides a syntactic criterion for the existence of the union of two timed scenarios. So the exact union can be computed when it exists. Otherwise, the combination (i.e., the “quasi-union”) provides an approximation for the exact union.

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