MATH 565 Spring 2019 - Class Notes

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Summary: This lecture focused on Chapter 8 in the textbook. We covered elementary properties of $\pi(x)$ as well as Chebyshev's Theorem.

Task 36: Proof of Euler's Observation

Proof. Suppose $n+1$ is prime $p = n+1$, $p \nmid a$ and $p \nmid b$. Want to prove that $p = n+1$ divides $a^n - b^n = a^{p-1} - b^{p-1}$.

Using Fermat's Little Theorem:

$a^n \equiv a^{p-1} \equiv 1 \pmod p$

$b^n \equiv b^{p-1} \equiv 1 \pmod p$

so $a^n - b^n \equiv a^{p-1} - b^{p-1} \equiv 1 - 1 \equiv 0 \pmod p$

Thus p divides $a^n - b^n$.

□

"How many prime numbers are there?"

$$\pi(x) = \#\{p \leq x \mid p \text{ is prime}\}$$
$$\pi(x) = \sum_{p \leq x} 1$$

Example 1: $\pi(10) = 4$; 2, 3, 5, 7

Example 2: $\pi(11) = 5$; 2, 3, 5, 7, 11

Example 3: $\pi(11.7) = 5$; 2, 3, 5, 7, 11

How is the function $\pi(x)$ growing?

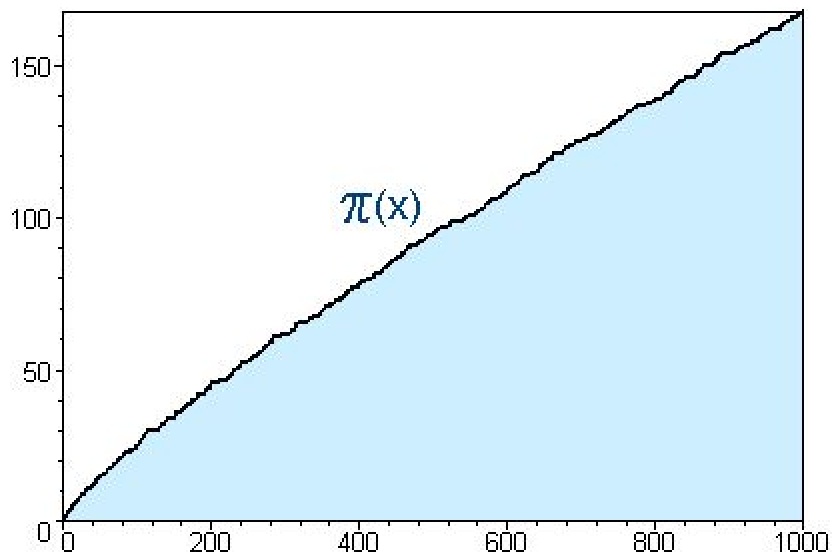

Figure 1. The number of primes less than or equal to $\pi(x)$.

Theorem 8-1:

$\lim_{x\to\infty} \pi(x) = \infty$ that is, there are infinitely many primes.

Proof. Take any list of prime numbers $\mathrm{p}_1, p_2, \cdots, p_k$.

Let n = $\mathrm{p}_1 \cdot p_2, \cdot p_3 \cdot \cdots \cdot p_k$.

Now consider n + 1

It is not divisible by any of $p_1, p_2, \cdots, p_k$ so by the Fundamental Theorem of Arithmetic, since n + 1 factors into primes, there has to be a prime not in this list. □

Theorem 8-2:

$\frac{\pi(x)}{x} < \frac{phi(k)}{k} + \frac{2k}{k}$ for any integer k.

Proof. Write $x = k\ell + r$ where $\ell = \left\lfloor \frac{x}{k} \right\rfloor$ and $r < k$.

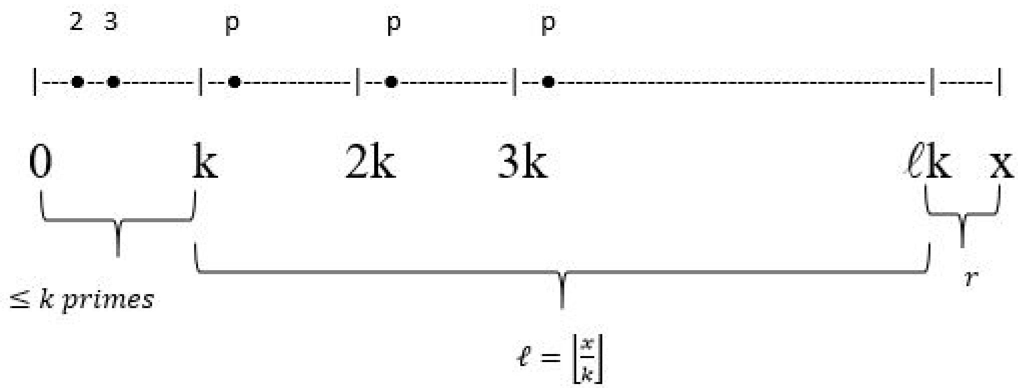

Figure 2. $\pi(x)$ counts the prime numbers to x.

In the first interval from 1 to k there are k integers so there can't be more than k prime numbers. Now in every other interval, of length k, there are $\phi(k)$ many integers coprime to k so there are at most $\phi(k)$ primes.

In the last remainder of size r, there are at most $r < k$ primes:

$\pi(x) \leq (k) + (\ell - 1)\phi(k) + r$

$\frac{\pi(x)}{x} \leq \frac{k}{x} + \frac{(\ell-1)}{x}\phi(k) + \frac{r}{x}$

$\frac{\pi(x)}{x} \leq \frac{k}{x} + \frac{\phi(k)}{x} + \frac{k}{x}$

$\frac{\pi(x)}{x} \leq \frac{\phi(k)}{x} + \frac{2k}{x}$

□

Theorem 8-3:

$$\sum_{n=1}^{M} \frac{1}{n} < \prod_{p<M} \left(\frac{1}{1-\frac{1}{p}} \right)$$

Equivalently, if $p_1, p_2, \cdots,$ p$_k$ are the primes less than M, then

$$\sum_{n=1}^{M} \frac{1}{n} < \left(\frac{1}{1-\frac{1}{p_1}} \right) \left(\frac{1}{1-\frac{1}{p_2}} \right) \cdots \left(\frac{1}{1-\frac{1}{p_k}} \right)$$

Proof. Note that if $n < M$, then all of the prime divisors of n are less than M. Now consider $1 - \frac{1}{p} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots$

$$\prod_{p<M} \left(1 - \frac{1}{p}\right) = \left(\frac{1}{1-\frac{1}{p_1}} \right) \left(\frac{1}{1-\frac{1}{p_2}} \right) \cdots \left(\frac{1}{1-\frac{1}{p_k}} \right)$$

$$= \left(1 + \frac{1}{p_1} + \frac{1}{p_2} + \cdots \right) \left(1 + \frac{1}{p_2} + \frac{1}{{p_2}^2} + \cdots \right) \cdots \left(1 + \frac{1}{p_k} + \frac{1}{{p_k}^2} + \cdots \right)$$

When you foil this out, you get every term that looks like $\frac{1}{{p_1}^{k_1}\ {p_2}^{k_2}\ {p_3}^{k_3 \cdots}}$

Since every integer $n < M$ shows up as a denominator in this foiled out product,

$$\sum_{n=1}^{M} \frac{1}{n} < \prod_{p<M} \left(\frac{1}{1-\frac{1}{p}} \right)$$

□

Example 4: Try this with M = 6

$$\sum_{n=1}^{6} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{2 \cdot 3} \quad \text{claim this is less than}$$

$$\prod_{p<6} \left(\frac{1}{1-p} \right) = \left(\frac{1}{1-\frac{1}{2}} \right) \left(\frac{1}{a-\frac{1}{3}} \right) \left(\frac{1}{1-\frac{1}{5}} \right)$$

$$= \left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{4} + \cdots \right) \left(1 + \frac{1}{3} + \frac{1}{9} + \cdots \right) \left(1 + \frac{1}{5} + \cdots \right)$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{3} + \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{4}\left(\frac{1}{3}\right) + \cdots + \frac{1}{5} + \frac{1}{2}\left(\frac{1}{5}\right)$$

Theorem 8-4:

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = 0$$

Goal: Prove that $\frac{pi(x)}{x} < \epsilon$ for any $\epsilon < 0$.

Proof. We know $\frac{pi(x)}{x} \leq \frac{\phi(k)}{k} + \frac{2k}{x}$. If we pick $p_1 \cdot p_2 \cdot \ldots \cdot p_\ell$,

$$\phi(k) = \phi(p_1)\phi(p_2)\cdots\phi(p_\ell)$$
$$= (p_1 - 1)(p_2 - 1)(p_3 - 1)\cdots(p_\ell - 1)$$

$$\frac{\phi(k)}{k} = \frac{(p_1-1)(p_2-1)(p_3-1)\cdots(p_\ell-1)}{p_1 \cdot p_2 \cdot \ldots \cdot p_\ell}$$
$$= \frac{p_1\left(1-\frac{1}{p_1}\right)p_2\left(1-\frac{1}{p_2}\right)\cdots p_\ell\left(1-\frac{1}{p_\ell}\right)}{p_1 \cdot p_2 \cdot \ldots \cdot p_\ell}$$
$$= \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_\ell}\right)$$

If $p_1 p_2 \cdots p_\ell$ are the primes less than M,

then $$\frac{\phi(k)}{k} = \prod_{p<M}\left(1 - \frac{1}{p}\right) < \left(\sum_{n=1}^{M} \frac{1}{n}\right)^{-1}$$

$$\sum_{n=1}^{M} \frac{1}{n} < \prod_{p<M}\left(1 - \frac{1}{p}\right)$$

$$\left(\sum_{n=1}^{M} \frac{1}{n}\right)^{-1} > \prod_{p<M}\left(1 - \frac{1}{p}\right)$$

By picking M to be big enough, we can make $\left(\sum_{n=1}^{M} \frac{1}{n}\right)^{-1} < \frac{\epsilon}{2}$ for any ϵ

If we take $k = p_1 p_2 \cdot p_\ell$, where p_ℓ is large enough so this term is less than $\frac{\epsilon}{2}$, then $\frac{phi(k)}{k} < \frac{\epsilon}{2}$.

$$\frac{\pi(x)}{x} < \frac{\phi(k)}{k} + \frac{2k}{x}$$

$$\frac{\phi(x)}{x} < \frac{\epsilon}{2} + \frac{2k}{x} \text{ for any x.}$$

Suppose $x > \frac{4}{\epsilon} p_1 p_2 \cdots p_\ell$,

then $\frac{\pi(x)}{x} < \frac{\epsilon}{2} + \frac{2p_1p_2\cdots p_\ell}{\frac{4}{\epsilon}p_1p_2\cdots p_\ell}$

$$\frac{\pi(x)}{x} < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\frac{\pi(x)}{x} < \epsilon$$

□

Note: **Theorem 8-4** tells us that the proportion of integers that are prime tends to be 0, thus 0% of integers are prime.

Prime Number Theorem

$$\lim_{x \to} \frac{\pi(x)}{\frac{x}{log(x)}} = 1; \qquad \text{where } \pi(x) \approx \frac{x}{log(x)}$$

Proof. Involves Complex Analysis (Hard Proof - not shown) □

Chebyshev's Theorem:

There exist positive constants c_1 and c_2, where $\frac{c_1 x}{logx} < \pi(x) < \frac{c_2 x}{logx}$
Note: $(log = ln = log_e)$

Facts that we need:

- $0 \leq \lfloor 2x \rfloor - \lfloor x \rfloor \leq 1$ **[Theorem 8-5]**
- $f(x) = \frac{x}{logx}$ then f(x) is increasing if $x > e$ and $f(x-2) > \frac{1}{2}f(x)$
- The exponent of the prime p in n! is equal to $\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p_i} \right\rfloor$

  $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)n$

  The number of these integers divisible by p is $\left\lfloor \frac{n}{p} \right\rfloor$.

  The number divisible by p^2 is $\left\lfloor \frac{n}{p^2} \right\rfloor$.

Example 5: What is the exponent on 3 in 11!?

$11! = 1 \cdot 2 \cdot \underline{3} \cdot 4 \cdot 5 \cdot \underline{6} \cdot 7 \cdot 8 \cdot \underline{9} \cdot 10 \cdot 11$

$$4 = \sum_{i=1}^{\infty} \left\lfloor \frac{11}{3^i} \right\rfloor = \left\lfloor \frac{11}{3} \right\rfloor + \left\lfloor \frac{11}{9} \right\rfloor + \left\lfloor \frac{11}{27} \right\rfloor$$

$$= 3 + 1$$

$$= 4$$

Proof. (Lower Bound for Chebyshev's Theorem)

Consider $\binom{2n}{n} = \frac{(2n)!}{n!n!}$

$= \frac{2n(2n-1)\cdots(n+1)}{n(n-1)\cdots 1}$

We know $\binom{2n}{n}$ is an integer.

If p is a prime between n and 2n, then p appears only in the numerator so $p \mid \binom{2n}{n}$

If we take any prime $p < 2n$, let r_p be the largest power of p less than 2n.

$p^{r_p} \leq 2n < p^{r_p+1}$

Example 6: If n = 5, 2n = 10, what is r_3?

$r_3 = 2 \qquad 3^2 \leq 10 < 3^{2+1}$

What is the power of p in $\binom{2n}{n} = \frac{(2n)!}{n!n!}$?

Using our formula for factorials this power is:

$$\sum_{i=1}^{\infty} \left\lfloor \frac{2n}{p_i} \right\rfloor - 2\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p_i} \right\rfloor$$

$$= \sum_{i=1}^{\infty} \left(\left\lfloor \frac{2n}{p_i} \right\rfloor - 2\left\lfloor \frac{n}{p_i} \right\rfloor \right)$$

$$= \sum_{i=1}^{r_p} \left(\left\lfloor \frac{2n}{p_i} \right\rfloor - 2\left\lfloor \frac{n}{p_i} \right\rfloor \right) \leq \sum_{i=1}^{r_p} 1 = r^p$$

Note: $(\lfloor 2x \rfloor - 2\lfloor x \rfloor) \leq 1$

p^{r_p} is the biggest power of p that could show up in $\binom{2n}{n}$.

Let $Q_n = \prod_{p<2n} p^{r_p}$ so $\binom{2n}{n} \mid Q_n$

Since $p^{r_p} \leq 2n$ and there are $\pi(2n)$ many primes in the product for Q_n, then $Q_n \leq (2n)^{\pi(2n)}$.

So, $\binom{2n}{n} \leq (2n)^{\pi(2n)}$

$$\binom{2n}{n} = \frac{2n(2n-1)(2n-2)\cdots(n+1)}{n(n-1)(n-2)\cdots 1}$$

$$= \left(\frac{2n}{n}\right)\left(\frac{2n-1}{n-1}\right)\cdots\left(\frac{n+1}{1}\right)$$

$$> (2)(2)(2)\cdot \ldots \cdot 2$$

$$\binom{2n}{n} > 2^n$$

So, $2^n < \binom{2n}{n} \leq (2n)^{\pi(2n)}$

$$2^n \leq (2n)^{\pi(2n)}$$

$$log(2^n) \leq log\left((2n)^{\pi(2n)}\right)$$

$$nlog2 \leq \pi(2n)log(2n)$$

So, $\frac{nlog2}{2n} \leq \pi(2n)$

Note: Lower End of Chebyshev's Theorem $\frac{c_1 k}{logx} < \pi(x)$

□

Works Cited

Caldwell, C. K. (n.d.). How Many Primes Are There? Retrieved April 28, 2019, from https://primes.utm.edu/howmany.html