MATH 565 Spring 2019 - Class Notes

4/3/19

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Summary: The lecture showed us how to estimate

$$\sum_{n\leq N} d(n)$$

Notes:
Recall the divisor function:

$$d(n) = \text{the number of divisors of n}$$

How big is $d(n)$ on average?
In other words, what is

$$\frac{1}{x}\sum_{n\leq x} d(n)$$

We'll discuss the **Euler Mascheroni constant**, written as γ.

Let's look at the **harmonic series**

$$\sum_{n\leq x} \frac{1}{n}$$

Does the series converge? No.

What does it mean to diverge?

$$\lim_{x\to\infty} \sum_{n\leq x} \frac{1}{n} = \infty$$

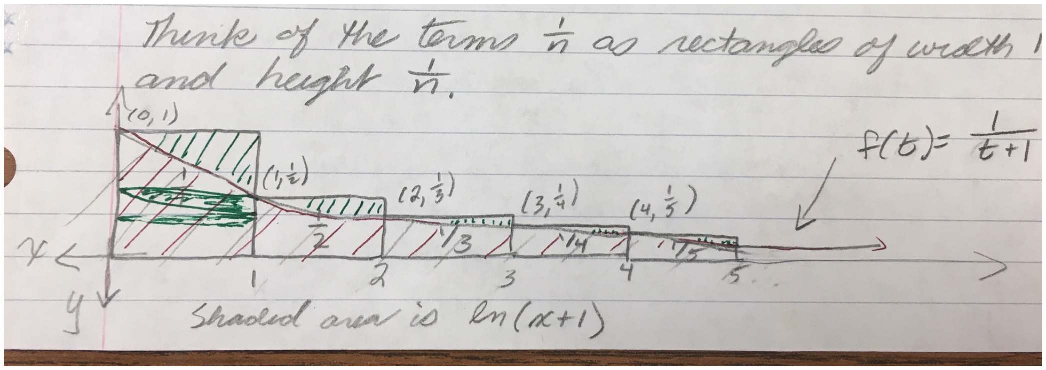
Let's approximate the sum

$$\sum_{n\leq x} \frac{1}{n}$$

.

We want a formula in terms of x that approximates the sum.

We think of the terms $\frac{1}{n}$ as rectangles of width 1 and height $\frac{1}{n}$.



Estimating the size of

$$\sum_{n \leq x} \frac{1}{n}$$

is the same as adding together the area of all of these rectangles up to x.

$$\sum_{n \leq x} \frac{1}{n} \approx \int_0^x \frac{1}{t+1} dt = \ln \mid t+1 \mid \Big|_{t=0}^{x}$$
$$= \ln x + 1$$

Take all of the little green slivers from every rectangle out to infinity. They can slide over into the first box. Add up the areas of all these small regions and that area is the constant $\gamma \approx 0.61$.

We think γ is irrational, but here is no proof.

Define

$$\gamma = \lim_{x \to \infty} (\sum_{n \leq x} \frac{1}{n} - \ln (x+1))$$

$$\sum_{n \leq x} \frac{1}{n} = \ln (x+1) + \gamma + E$$

where $\mid E \mid < \frac{1}{x}$. E is the error term which represents all the green space in the picture. As x gets larger (approaches ∞), E gets smaller (approaches 0).

Big Oh Notation is a better way to describe this error E. Define

$$f(x) = O(g(x))$$

to mean that $\mid f(x) \mid \leq kg(x)$ for some constant k and large enough x. Examples:

- $x = O(x^2)$
- $3x = O(x)$

- $sin(x) = O(x)$

Prove that $sin(x) = O(x)$.

$\mid sin(x) \mid \leq 1 * 1$, where the first 1 is k and the second 1 is g(x).

Also, $\mid sin(x) \mid \leq 5 * 1$

$x^2 \neq O(x)$

If $x^2 = O(x)$ were true, there would have to be a k such that $x^2 \leq kx$. $ln(x) = O(x)$ because $ln(x) < x$ for all x. In general, $x^a = O(x^b)$ if $b \leq a$.

$\sqrt{x} = x^{1/2} = O(x^{2/3})$

We can use Big Oh notation with other expressions like:

- $x^3 + 3x + 1 + cos(x) = x^3 + O(x)$

  x^3 is the main term.

  $3x + 1 + cos(x) = O(x)$ because $3x + 1 + cos(x)$ isn't going to grow faster than some constant times x.

- $3x^2 + \frac{1}{x} = 3x^2 + O(1)$

  $\frac{1}{x} = O(1)$

So our sum

$$\sum_{n \leq x} \frac{1}{n} = \ln(x+1) + \gamma + E$$

becomes

$$\sum_{n \leq x} \frac{1}{n} = \ln(x+1) + \gamma + O(\frac{1}{n})$$

and, less precisely, but also true:

$$\sum_{n \leq x} \frac{1}{n} = \ln(x+1) + O(1)$$

<u>Goal:</u> Estimate

$$\sum_{n \leq N} d(n)$$

$$\sum_{n \leq N} (\sum_{d|n} 1) = \sum_{n \leq N} (\sum_{de=n} 1)$$

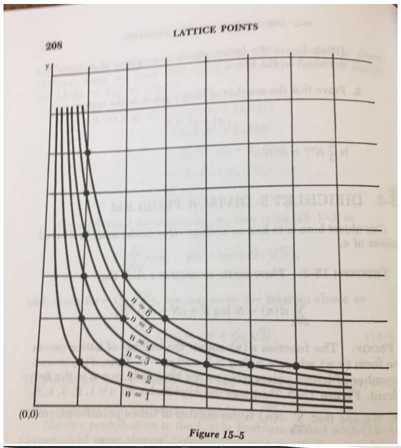
Note:

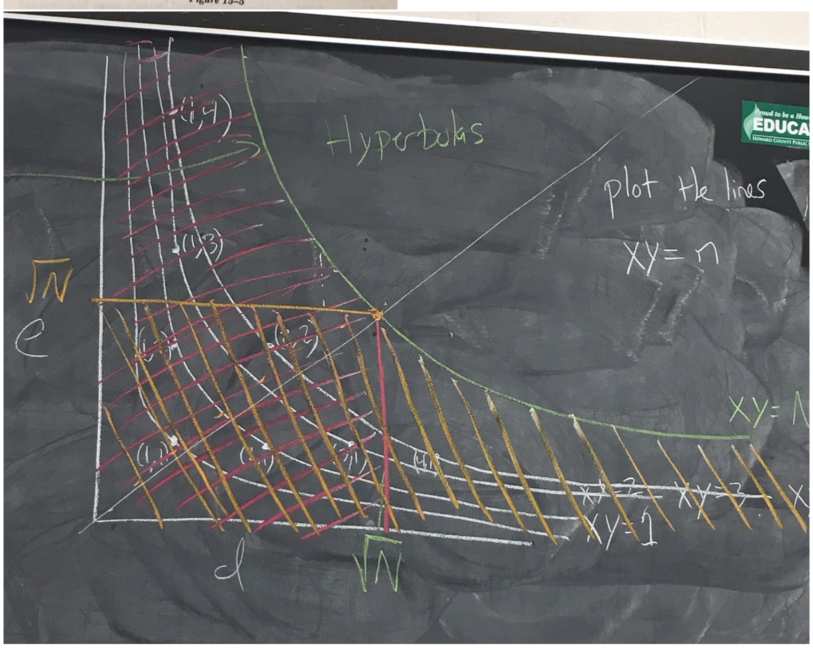
$$\sum_{n \leq N} (\sum_{de=n} 1)$$

counts all the ways to find two numbers d and e so that $de \leq x$.

$$\sum_{n\leq N}(\sum_{de=n} 1) = \sum_{de\leq N} 1$$

counts all the lattice points below the hyperbola xy=n.





$$\sum_{de=n} 1 = \sum_{d<\sqrt{N}}(\sum_{e\leq\frac{\sqrt{N}}{d}} 1) + \sum_{e<\sqrt{N}}(\sum_{d\leq\frac{\sqrt{N}}{e}} 1) - \lfloor\sqrt{N}\rfloor^2$$

$$= 2\sum_{d<\sqrt{N}}(\sum_{e\leq\frac{\sqrt{N}}{d}} 1) - \lfloor\sqrt{N}\rfloor^2$$

$$= 2\sum_{d<\sqrt{N}}\lfloor\frac{N}{d}\rfloor - \lfloor\sqrt{N}\rfloor^2$$

$$= 2\sum_{d<\sqrt{N}}(\frac{N}{d} - \{\frac{N}{d}\}) - \lfloor\sqrt{N}\rfloor^2$$

$$= 2\sum_{d<\sqrt{N}}\frac{N}{d} - 2\sum_{d<\sqrt{N}}\{\frac{N}{d}\} - \lfloor\sqrt{N}\rfloor^2$$

$$= 2\sum_{d<\sqrt{N}}\frac{N}{d} - O(\sqrt{N}) - \lfloor\sqrt{N}\rfloor^2$$

$$= 2N\sum_{d<\sqrt{N}}\frac{1}{d} - O(\sqrt{N}) - N - O(\sqrt{N})$$

$$= 2N\sum_{d<\sqrt{N}}\frac{1}{d} - N + O(\sqrt{N})$$

Recall:

$$\sum_{n\leq x}\frac{1}{n} = \ln{(x+1)} + \gamma + O(\frac{1}{n})$$

So continuing from above:

$$= 2N(\ln\sqrt{N} + 1) + \gamma + O(\frac{1}{\sqrt{N}} - N + O(\sqrt{N})$$

$$= 2N(\ln\sqrt{N}) + \gamma + O(\frac{1}{\sqrt{N}} - N + O(\sqrt{N})$$

$$= 2N(\frac{1}{2}\ln N) + \gamma + O(\frac{1}{\sqrt{N}} - N + O(\sqrt{N})$$

$$= N\ln N + 2N\gamma - N + O(\sqrt{N})$$

$$= N\ln N + (2\gamma - 1)N + O(\sqrt{N})$$

So the average size of d(n):

$$\frac{1}{N}\sum_{n\leq N}d(n) = \ln N + 2\gamma - 1 + O(\frac{1}{\sqrt{N}})$$