

## Math 565 - Spring 2019

### Homework 8

Due May 1st, 2019

*There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, despite their simple definition and role as the building blocks of the natural numbers, the prime numbers grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision.*

— Don Zagier

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- (1) For which prime  $p < 20$  is there the most primitive roots? (Hint: don't actually find the primitive roots...)
- (2) Using the information that 7 is a primitive root modulo 22, determine all integers modulo 22 that are primitive roots.
- (3) (a) Determine the order of 2 modulo  $2^n - 1$ .  
(b) Use this to prove that  $\varphi(2^n - 1)$  is divisible by  $n$ .
- (4) Modify Euclid's proof that there are infinitely many primes to show that there are infinitely many primes congruent to 3 (mod 4).
- (5) Problem 8-2.1