Math 490 - Spring 2016
Midterm Practice

In mathematics you don't understand things. You just get used to them.

— John von Neumann

Practice:

(1) Find the ordinary generating function for the following sequences:

(a) $a_n = 4$

$$\sum_{n=0}^{\infty} 4x^n = \frac{4}{1-x}$$

(b) $a_n = n + 1$

$$\sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n = \frac{x}{(1-x)^2} + \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{1}{(1-x)^2}$$

(c) $a_n = 3^n$

$$\sum_{n=0}^{\infty}(3x)^n = \frac{1}{1-3x}$$

(d) $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 1, a_1 = 1$.

$$\begin{aligned}
f(x) = \sum_{n=0}^{\infty} a_n x^n &= 1 + x + \sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} 2a_{n-2}x^n \\
&= 1 + x + x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 2x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} \\
&= 1 + x + x\sum_{n=1}^{\infty} a_n x^n + 2x^2\sum_{n=0}^{\infty} a_n x^n \\
&= 1 + x + x(f(x) - 1) + 2x^2 f(x) \\
&= 1 + xf(x) + 2x^2 f(x)
\end{aligned}$$

Solving for $f(x)$:

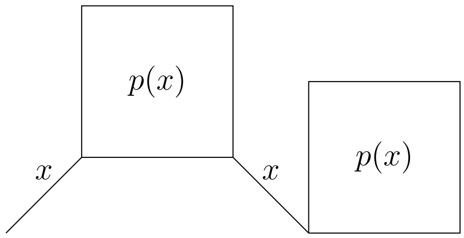
$$f(x) = \frac{1}{1 - x - 2x^2}$$

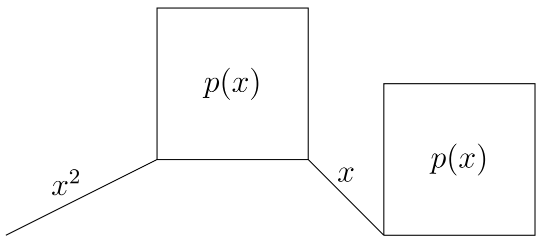
(2) Find a generating function for the number of Dyck-Path-like walks that consist either of

- Up-Steps of slope 1, (length 1)
- Half-Up-Steps of slope 1/2 (length 2)
- Down-Steps of slope -1 (Length 1)

Hint: The sequence begins: 0,1,1,2,4,7... Draw pictures!

Each path begins with either an up step of slope 1 or an up step of slope 1/2. Then just like with Dyck paths it must at some point return to the x-axis with a down step. Thus each path can be counted by either:





or the empty path. Thus

$$p(x) = 1 + x^2 p(x)^2 + x^3 p(x)^2.$$

Solving for $p(x)$, we get

$$0 = 1 - p(x) + (x^2 + x^3)p(x)^2$$

and so

$$p(x) = \frac{1 \pm \sqrt{1 - 4x^2 - 4x^3}}{2x^2 + 2x^3}.$$

As usual, we take the root corresponding to the minus sign to obtain a generating function.

(3) For a fixed integer $k \geq 0$, find the exponential generating function for $\left\{ \binom{n}{k} \right\}_{n=0}^{\infty}$

Since $\binom{n}{k} = 0$ when $n < k$, we get that the exponential generating function for this sequence is

$$\sum_{n=0}^{\infty} \binom{n}{k} \frac{x^n}{n!} = \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} \left(\frac{x^n}{n!} \right) = \sum_{n=k}^{\infty} \frac{x^n}{k!(n-k)!}$$
$$= \frac{x^k}{k!} \sum_{n=k}^{\infty} \frac{x^{n-k}}{(n-k)!} = \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{x^k e^x}{k!}$$

(4) Suppose $f(n)$ is a function that satisfies $n^2 = \sum_{d|n} f(d)$. Use Möbius inversion to find a formula for $f(n)$, and use it to compute the values of $f(n)$ for $1 \leq n \leq 6$.

By Möbius inversion,

$$f(n) = \sum_{d|n} d^2 \mu \left(\frac{n}{d} \right).$$

Using this we compute $f(1) = 1$, $f(2) = 3$, $f(3) = 8$, $f(4) = 14$, $f(5) = 24$, $f(6) = 24$.