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Rule for multiplying Ordinary Generating Function (OGF) was:

f(x) is OGF $[a_n]_{n=0}^{\infty}$

g(x) is OGF $[b_n]_{n=0}^{\infty}$

then,

f(x)g(x) is OGF of $[\sum_{k=0}^{n} a_k b_{n-k}]_{n=0}^{\infty}$

Rule for multiplying Exponential Generating Function (EGF) was:

f(x) is EGF $[a_n]_{n=0}^{\infty}$

g(x) is EGF $[b_n]_{n=0}^{\infty}$

then,

f(x)g(x) is EGF of $[\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}]_{n=0}^{\infty}$

Exponential Generating Function (EGF) of Derangements:

D(x)=$\frac{e^{-x}}{1-x}$=$(e^{-x})(\frac{1}{1-x})$=$(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(x^n))(\sum_{n=0}^{\infty} \frac{n!}{n!}(x^n))$

D(x) is the EGF of :

$[\sum_{k=0}^{n} \binom{n}{k}(-1)^k(n-k)!]_{n=0}^{\infty}$

$[n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}]_{n=0}^{\infty}$

Definition: If a_n is a sequence, then the Dirichlet Series Generating Function (DSGF) of $[a_n]_{n=0}^{\infty}$ is:

f(s)=$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$

f(s) is DSGF of $[a_n]$
g(s) is DSGF of $[b_n]$
What is f(s)g(s)?

$(\frac{a_1}{1^s} + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + ...)(\frac{b_1}{1^s} + \frac{b_2}{2^s} + \frac{b_3}{3^s} + \frac{b_4}{4^s} + ...)$=

$(\frac{a_1b_1}{1^s}) + (\frac{a_1b_2+a_2b_1}{2^s}) + (\frac{a_1b_3+a_3b_1}{3^s}) + ... + (\frac{a_1b_4+a_2b_2+a_4b_1}{4^s}) + ...$

Rule for multiplying Dirichlet Series Generating Function (DSGF) is:

f(s) is DSGF of $[a_n]$

g(s) is DSGF of $[b_n]$

then,

f(s)g(s) is DSGF of $[\sum_{d|n} a_d b_{n/d}]_{n=0}^{\infty}$

Dirichlet Series Generating Function makes sense when objects of size n are made by stitching together objects of k with size $\frac{n}{k}$.

The OGF of $[1]_{n=0}^{\infty}$ was

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

The EGF of $[1]_{n=0}^{\infty}$ was

$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$

The DSGF of $[1]_{n=0}^{\infty}$ is $\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$

$\Rightarrow \zeta(s)$ is the Riemann Zeta Function.

$\Rightarrow$ A special case of Riemann Zeta Function is: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

What is $\zeta(s)\zeta(s)$?

This is the DSGF for the sequence:

$[\sum_{d|n} (1)(1)]_{n=1}^{\infty}$

d(n)=$[\sum_{d|n} 1]$ = the number of divisors of n.

The sequence: $[d(n)]_{n=1}^{\infty} = (1, 2, 2, 3, 2, 4, ...)$ The Dirichlet Series Generating Function for the number of divisors of n is:

$(\frac{1}{1^s} + \frac{2}{2^s} + \frac{2}{3^s} + \frac{3}{4^s} + \frac{2}{5^s} + \frac{4}{6^s} + ...) = \zeta(s)^2$

<u>Definition:</u> A function is *number theoretic* if its domain is the positive integers.

Definition: A number-theoretic function f(n) is *multiplicative* if f(mn)=f(m)f(n), whenever gcd(m,n)=1.

If f(n) is multiplicative, then

$$f(p_1^{a_1}p_2^{a_2}...p_k^{a_k}) = f(p_1^{a_1})f(p_2^{a_2})...f(p_k^{a_k})$$

A multiplicative function is determined by its values on the prime powers.

Example Let $f(p^k) = (p^k)^2 = p^{2k}$

Then, if we want to compute f(n), write $[n = p_1^{a_1}p_2^{a_2}...p_k^{a_k}]$

f(n) $= f(p_1^{a_1})f(p_2^{a_2})...f(p_k^{a_k})$

$=p_1^{2a_1}p_2^{2a_2}...p_k^{2a_k}$

$=(p_1^{a_1}p_2^{a_2}...p_k^{a_k})^2$

$=n^2$

So, n is a multiplicative function.

d(n) is also a multiplicative function.

$d(12) = 6$

$12 = 2^2 * 3$

$d(2^2) = 3$

$d(3) = 2$

$d(12) = d(2^2) * d(3) = 3 * 2.$

Why is this true??

n=lk

gcd(lk)=1

Let d be a divisor of n. All of the prime factors of d appear in l and k.

So, $d = d'd''$, where $d'|l$ and $d''|k$.

This is unique.

Every divisor of n comes from combining exactly one divisor of l with one divisor of k.

So, the total number of divisors of n is the product of the numbers of divisors of l with the number of divisors of k.

<u>Theorem:</u> If f(n) is a multiplicative function then,

$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p [1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} + ...]$

<u>Proof:</u>

$[1 + \frac{f(2)}{2^s} + \frac{f(2^2)}{2^{2s}} + \frac{f(2^3)}{2^{3s}} + ...] * x[1 + \frac{f(3)}{3^s} + \frac{f(3^2)}{3^{2s}} + \frac{f(3^3)}{3^{3s}} + ...]$

What is the term with a denominator of 24^s?

$\Rightarrow$ Only way to produce a 24^s term in the denominator is to "choose" the 2^{-3s} term when multiplying $p = 2$ and the 3^{-s} when multiplying $p = 3$ and the 1 for every higher prime.

Numerator of this term has to be:

$f(2^3)f(3) = f(24)$, by multiplicativity of f(n).

By equating coefficients, we see that the only way to get a denominator of n^{-s} is by multiplying together the terms corresponding to the Prime Factorization of n.

What does this say for $\zeta(s)$??

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

$= \prod_p [1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + ...]$

$= \prod_p [\sum_{i=0}^{\infty} [\frac{1}{p^s}]^i]$

$= \prod_p [\frac{1}{1 - \frac{1}{p^s}}]$

$= \prod_p [\frac{1}{1 - p^{-s}}] \Rightarrow$ "Euler Product"

Lets make a new multiplicative function by defining its values on prime powers declare it to be multiplicative.

$$\mu(p^k) = \begin{cases} 1, & \text{if } k = 0 \\ -1, & \text{if } k = 1 \\ 0, & \text{if } k > 1 \end{cases}$$

$\mu(2) = -1$

$\mu(3) = -1$

$\mu(4) = 0 = \mu(2^2)$

$\mu(6) = \mu(2)\mu(3) = (-1)(-1) = 1$

This function is called the "Mobius Function."

$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$

$= \prod_p (1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + ...)$

$= \prod_p (1 - \frac{1}{p^s})$