LECTURE NOTES 3/7/17

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Let A, B, and C be non-disjoint sets. We need to find a way to count the size of $A \cup B \cup C$ without counting things in multiple sets more than once. To do this, we subtract off things that show up multiple times.

$$|A \cup B \cup C| = (|A| + |B| + |C|) - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

This is generalized by:

Principle of Inclusion Exclusion

If $A = A_1 \cup A_2 \cup ... \cup A_n$, then the size of A is

$$|A| = \sum_{i=1}^{n} \sum_{\substack{J \subset \{1,2,...,n\} \\ |J|=i}} (-1)^{i+1} \left| \bigcap_{j \in J} A_j \right|$$

Counting Derangements

Suppose n people leave their hats on a table, and then everyone takes a hat. What is the probability that no one gets their own hat back?

We can describe this situation using permutations. Let a permutation σ denote which hat the ith person got.

Example. $\sigma = 3\,2\,1\,4$
Both person 2 and person 4 got their own hats. Persons 1 and 3 swapped hats.

Every permutation of length n corresponds to a way the hats could be mixed up.
The probability that no one gets their own hat is

$$\frac{\#\text{ of permutations where } \sigma(i) \neq i \,\forall i}{\#\text{ of permutations of length } n}$$

A **derangement** is a permutation σ where $\sigma(i) = i \,\forall i$.

We need to count the derangements of n.
Let $A_i = \{\text{permutations of length } n \text{ where } \sigma(i) = i\}$,
$D_n = \{\text{derangents of } n\}$,
$S_n = \{\text{permutations of length } n\}$, and

$A = A_1 \cup A_2 \cup ... \cup A_n$.

Then,

$D_n = S_n \backslash A \qquad \text{and} \qquad |D_n| = n! - |A|$.

$|A_i| = (n-1)!$ since we fix i and permute the other $n-1$.

$A_i \cap A_j| = (n-2)!$ since we fix i and j and permute the other $n-2$.

In general,

$$\left| \bigcap_{\substack{j \in J \\ |J|=k}} A_j \right| = (n-k)!$$

Apply the principle of inclusion exclusion

$$\begin{aligned} |A| &= \sum_{i=1}^{n} \sum_{\substack{J \subset \{1,2,...,n\} \\ |J|=i}} (-1)^{i+1} \left| \bigcap_{j \in J} A_j \right| \\ &= \sum_{i=1}^{n} \sum_{\substack{J \subset \{1,2,...,n\} \\ |J|=i}} (-1)^{i+1} (n-i)! \\ &= \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (n-i)! \\ &= \sum_{i=1}^{n} (-1)^{i+1} \frac{n!}{i!} \end{aligned}$$

So,

$$\begin{aligned} |D_n| &= n! - n! \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!} \\ &= n! \sum_{i=1}^{n} \frac{(-1)^i}{i!} \\ &= n! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + \frac{(-1)^n}{n!} \right) \end{aligned}$$

Therefore, the probability that no one gets their own hat is

$$\text{Probablility} = \frac{|D_n|}{|S_n|} = \frac{n! \sum_{i=1}^{n} \frac{(-1)^i}{i!}}{n!} = \sum_{i=1}^{n} \frac{(-1)^i}{i!}$$

$$\approx e^{-1} \approx 0.368 \text{ when } n \text{ is large}$$

Exponential Generating Function for Permutations

A permutation of n is a sequence of labelled single vertices.

Example. $\sigma = 2\,4\,1\,3\,5$
The number in the ith position can be seen as the label of the ith single vertex.

The Exponential generating function for single vertices is just

$$V(x) = x$$

since there is only 1 vertex and it has size 1.
Since a permutation is a sequence of labelled single vertices, the EGF for permutations is

$$\begin{aligned} P(x) &= (1 + V(x) + V(x)^2 + V(x)^3 + ...) \\ &= \frac{1}{1 - V(x)} \\ &= \frac{1}{1 - x} \end{aligned}$$

Another way to write a permutation is in cycle notation.
For example, consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$.
In cycle notation, $\sigma = (1\,3\,5\,4)(2)(6)$.

A permutation is a set of cycles. The order of the cycles doesn't matter.
We need to count cycles of length n.
Cycle $c = (a_1, a_2, a_3, ..., a_n)$, where the a_i's are numbers from 1 to n.
Cycles stay the same when rotated.

Example. $(1\ 3\ 4\ 2) = (2\ 1\ 3\ 4) = (4\ 2\ 1\ 3) = (3\ 4\ 2\ 1)$

By fixing the choice of 1 in the 1st position, we see that there are $(n-1)!$ cycles of length n.
The EGF for cycles is

$$\begin{aligned} C(x) &= \sum_{n=1}^{\infty} \frac{(n-1)!}{n!} x^n \\ &= \sum_{n=1}^{\infty} \frac{1}{n} x^n \\ &= \ln\left(\frac{1}{1-x}\right) \end{aligned}$$

A permutation is a labelled set of cycles, so the Exponential generating function for permutations is

$$\begin{aligned}P(x) &= \left(1 + C(x) + \frac{C(x)^2}{2!} + \frac{C(x)^3}{3!} + ...\right)\\ &= \exp(C(x))\\ &= \exp\left(\ln\left(\frac{1}{1-x}\right)\right)\\ &= \frac{1}{1-x}\end{aligned}$$

Counting Derangements using EGFs

A derangement is a permutation with no cycles of length 1. In other words, it is a set of cycles all of length greater than 1.
The EGF for derangements is

$$\begin{aligned}D(x) &= \exp(C(x) - x)\\ &= \exp\left(\ln\left(\frac{1}{1-x}\right) - x\right)\\ &= \frac{e^{-x}}{1-x}\end{aligned}$$

Recall: If $f(x)$ and $g(x)$ are EGF's,

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$$

Then,

$$[x^n]f(x)g(x) = \sum_{n=0}^{n} \binom{n}{i} a_i b_{n-i}$$

Let $f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$ and $g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Then,

$$\begin{aligned}[x^n](e^{-x})\left(\frac{1}{1-x}\right) &= \sum_{n=0}^{n} \binom{n}{i} (-1)^i (n-i)!\\ &= n! \sum_{i=1}^{n} \frac{(-1)^i}{i!}\end{aligned}$$