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Dirichlet Series Generating Functions

Marissa Whitby

1 Dirichlet Series Generating Functions

- **General form for a Dirichlet Series Generating Function (dsgf)**:

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- **All 1's Dirichlet Series Generating Function (dsgf)**:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

- **Möbius Dirichlet Series Generating Function (dsgf)**:

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

- We know that as long as the sequence is multiplicative we can break it into its Euler Product.

Example 1:

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}$$

Define

$$\mu(p^k) = \begin{cases} 1 & \text{if } k = 0 \\ -1 & \text{if } k = 1 \\ 0 & \text{if } k \leq 2 \end{cases}$$

Declare $\mu(n)$ to be multiplicative on all integers.

Find $\mu(30)$:

$$\mu(30) = \mu(5 \cdot 3 \cdot 2)$$
$$= \mu(3) \cdot \mu(5) \cdot \mu(2)$$
$$= (-1)^3 = -1$$

Find $\mu(12)$:

$$\mu(12) = \mu(2^2 \cdot 3)$$
$$= \mu(2^2) \cdot \mu(3)$$
$$= -1 \cdot 0 = 0$$

So we see that μ is multiplicative and we can write it out using the Euler Product.

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \frac{\mu(p^3)}{p^{3s}} + ...)$$

However, we know that all $\mu(p^k) = 0$ for $k \geq 2$. Also, we know that $\mu(p) = 1$

So,

$$M(s) = \prod_p (1 - \frac{1}{p^s})$$

2 Deriving the Möbius Inversion Formula

- What happens when we multiply $\zeta(s)$ and $M(s)$?

First multiply them by using the dsgf's for the two functions:

$$\zeta(s)M(s) = (\sum_{n=1}^{\infty} \frac{1}{n^s}) \cdot (\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s})$$

$$= \sum_{n=1}^{\infty}((\sum_{d|n}\mu(d)) \cdot \frac{1}{n^s})$$

Now multiply them by using their Euler Products:

$$\zeta(s)M(s) = (\prod_p 1 - p^{-s}) \cdot (\prod_p \frac{1}{1-p^{-s}})$$

$$= \prod_p ((1-p^{-s}) \cdot \frac{1}{1-p^{-s}})$$

$$= \prod_p 1 = 1$$

So, in the land of Dirichlet Series Generating Functions these two functions are inverses.

- Now, consider the value of

$$\sum_{d|n}\mu(d)$$

which was obtained by multiplying the dsgf's of $\zeta(s)$ and $M(s)$. We know that it must be equal to the sequence that gives a 1 followed by all 0's so,

$$\sum_{d|n}\mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Compute the value of the sum when n=12

$$\sum_{d|12}\mu(d) = \mu(1) + \mu(2) + \mu(3) + \mu(4) + \mu(6) + \mu(12)$$

$$= 1 - 1 - 1 + 0 + 1 + 0$$
$$= 0$$

This will be true for all integers greater than 1.

- Suppose we have sequences a_n and b_n such that

$$a_n = \sum_{d|n} b_d$$

This is the formula for a_n given b_n. We want to find a formula for b_n in terms of a_n.

Let A(s) be the dsgf for a_n and let B(s) be the dsgf for b_n. The formula for a_n can be rewritten:

$$a_n = \sum_{d|n} b_d \cdot 1$$

Which tells us that $A(s) = B(s) \cdot \zeta(s)$
Therefore, to isolate B(s), multiply both sides of the equation by the inverse of $\zeta(s)$, which is M(s).
$M(s) \cdot A(s) = B(s) \cdot \zeta(s) \cdot M(s)$
$M(s) \cdot A(s) = B(s)$

So,

$$b_n = \sum_{d|n} a_d \cdot \mu(\frac{n}{d})$$

This is called the **Möbius Inversion Formula!**

3 Counting Primitive Bit Strings

- **Definition**: A **bit string** is a sequence of just 0's and 1's.
Example: 011011000

- **Definition**: We say a bit string is **primitive** if it cannot be written as the concatenation of identical smaller strings.
Examples:
010101 is not primitive because it is "01" concatenated together 3 times.
0100 is primitive.
1 is primitive.
11 is not primitive because it is "1" concatenated together 2 times.

- How many primitive bit strings are there of length n=4?

There are 2^4 total bit strings length 4.
They are:
~~0000~~, 0001, 0010, 0011, 0100, ~~0101~~, 0110, 0111, 1000, 1001, ~~1010~~, 1100, 1101, ~~1111~~, 1110, 1011

Note that all non-primitive bit strings have been crossed out. So, from this we see that for n=4 there are 12 primitive bit strings.

- How many primitive bit strings are there of a given length n?

  If a bit string is not primitive, then it is the concatenation of a string of length d where $d|n$ for n=length of the bit string.

  Given a bit string of length d, there is exactly one non-primitive bit string of length n formed by concatenating the string $\frac{n}{d}$ times.

  Every bit string of length n corresponds to exactly one primitive bit string of length d where $d|n$.

  So, there are 2^n total bit strings of length n and each correspond to 1 primitive bit string.

  Let f(n) count the number of primitive bit strings of length n.

$$2^n = \sum_{d|n} f(d)$$

  We need to solve the formula for f(d). By the Möbius Inversion Formula we get

$$f(n) = \sum_{d|n} 2^d \cdot \mu(\frac{n}{d})$$

- We can verify the formula for the number of primitive bit strings of length n by testing it with the n=4 case.
  For n=4:

$$f(4) = \sum_{d|4} 2^d \cdot \mu(\frac{4}{d})$$

$$\begin{aligned}
&= 2 \cdot \mu(4) + 2^2 \cdot \mu(\tfrac{4}{2}) + 2^4 \cdot \mu(\tfrac{4}{4}) \\
&= 2 \cdot \mu(4) + 2^2 \cdot \mu(2) + 2^4 \cdot \mu(1) \\
&= 0 - 2^2 + 2^4 \\
&= 12
\end{aligned}$$

Clearly, the formula for f(n) holds because this is exactly what we got when we calculated it the other way too.