

Senior Seminar Notes March 16th 2017

Dirichlet Series Generating Functions

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1 Dirichlet Series Generating Functions

- General form for a Dirichlet Series Generating Function (dsgf):

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

- All 1's Dirichlet Series Generating Function (dsgf):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

- Möbius Dirichlet Series Generating Function (dsgf):

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

- We know that as long as the sequence is multiplicative we can break it into its Euler Product.

Example 1:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

Define

$$\mu(p^k) = \begin{cases} 1 & \text{if } k = 0 \\ -1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases}$$

Declare $\mu(n)$ to be multiplicative on all integers.

Find $\mu(30)$:

$$\begin{aligned}\mu(30) &= \mu(5 \cdot 3 \cdot 2) \\ &= \mu(3) \cdot \mu(5) \cdot \mu(2) \\ &= (-1)^3 = -1\end{aligned}$$

Find $\mu(12)$:

$$\begin{aligned}\mu(12) &= \mu(2^2 \cdot 3) \\ &= \mu(2^2) \cdot \mu(3) \\ &= -1 \cdot 0 = 0\end{aligned}$$

So we see that μ is multiplicative and we can write it out using the Euler Product.

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \frac{\mu(p^3)}{p^{3s}} + \dots\right)$$

However, we know that all $\mu(p^k) = 0$ for $k \geq 2$. Also, we know that $\mu(p) = 1$

So,

$$M(s) = \prod_p \left(1 - \frac{1}{p^s}\right)$$

2 Deriving the Möbius Inversion Formula

- What happens when we multiply $\zeta(s)$ and $M(s)$?

First multiply them by using the dsgf's for the two functions:

$$\zeta(s)M(s) = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right)$$

$$= \sum_{n=1}^{\infty} \left(\sum_{d|n} \mu(d) \right) \cdot \frac{1}{n^s}$$

Now multiply them by using their Euler Products:

$$\begin{aligned} \zeta(s)M(s) &= \left(\prod_p (1 - p^{-s}) \right) \cdot \left(\prod_p \frac{1}{1 - p^{-s}} \right) \\ &= \prod_p \left((1 - p^{-s}) \cdot \frac{1}{1 - p^{-s}} \right) \\ &= \prod_p 1 = 1 \end{aligned}$$

So, in the land of Dirichlet Series Generating Functions these two functions are inverses.

- Now, consider the value of

$$\sum_{d|n} \mu(d)$$

which was obtained by multiplying the dsgf's of $\zeta(s)$ and $M(s)$. We know that it must be equal to the sequence that gives a 1 followed by all 0's so,

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Compute the value of the sum when $n=12$

$$\begin{aligned} \sum_{d|12} \mu(d) &= \mu(1) + \mu(2) + \mu(3) + \mu(4) + \mu(6) + \mu(12) \\ &= 1 - 1 - 1 + 0 + 1 + 0 \\ &= 0 \end{aligned}$$

This will be true for all integers greater than 1.

- Suppose we have sequences a_n and b_n such that

$$a_n = \sum_{d|n} b_d$$

This is the formula for a_n given b_n . We want to find a formula for b_n in terms of a_n .

Let $A(s)$ be the dsfg for a_n and let $B(s)$ be the dsfg for b_n . The formula for a_n can be rewritten:

$$a_n = \sum_{d|n} b_d \cdot 1$$

Which tells us that $A(s) = B(s) \cdot \zeta(s)$

Therefore, to isolate $B(s)$, multiply both sides of the equation by the inverse of $\zeta(s)$, which is $M(s)$.

$$M(s) \cdot A(s) = B(s) \cdot \zeta(s) \cdot M(s)$$

$$M(s) \cdot A(s) = B(s)$$

So,

$$b_n = \sum_{d|n} a_d \cdot \mu\left(\frac{n}{d}\right)$$

This is called the **Möbius Inversion Formula!**

3 Counting Primitive Bit Strings

- **Definition:** A **bit string** is a sequence of just 0's and 1's.

Example: 011011000

- **Definition:** We say a bit string is **primitive** if it cannot be written as the concatenation of identical smaller strings.

Examples:

010101 is not primitive because it is "01" concatenated together 3 times.

0100 is primitive.

1 is primitive.

11 is not primitive because it is "1" concatenated together 2 times.

- How many primitive bit strings are there of length $n=4$?

There are 2^4 total bit strings length 4.

They are:

0000, 0001, 0010, 0011, 0100, ~~0101~~, 0110, 0111, 1000, 1001, ~~1010~~,
1100, 1101, ~~1111~~, 1110, 1011

Note that all non-primitive bit strings have been crossed out. So, from this we see that for $n=4$ there are 12 primitive bit strings.

- How many primitive bit strings are there of a given length n ?

If a bit string is not primitive, then it is the concatenation of a string of length d where $d|n$ for n =length of the bit string.

Given a bit string of length d , there is exactly one non-primitive bit string of length n formed by concatenating the string $\frac{n}{d}$ times.

Every bit string of length n corresponds to exactly one primitive bit string of length d where $d|n$.

So, there are 2^n total bit strings of length n and each correspond to 1 primitive bit string.

Let $f(n)$ count the number of primitive bit strings of length n .

$$2^n = \sum_{d|n} f(d)$$

We need to solve the formula for $f(d)$. By the Möbius Inversion Formula we get

$$f(n) = \sum_{d|n} 2^d \cdot \mu\left(\frac{n}{d}\right)$$

- We can verify the formula for the number of primitive bit strings of length n by testing it with the $n=4$ case.
For $n=4$:

$$\begin{aligned} f(4) &= \sum_{d|4} 2^d \cdot \mu\left(\frac{4}{d}\right) \\ &= 2 \cdot \mu(4) + 2^2 \cdot \mu\left(\frac{4}{2}\right) + 2^4 \cdot \mu\left(\frac{4}{4}\right) \\ &= 2 \cdot \mu(4) + 2^2 \cdot \mu(2) + 2^4 \cdot \mu(1) \\ &= 0 - 2^2 + 2^4 \\ &= 12 \end{aligned}$$

Clearly, the formula for $f(n)$ holds because this is exactly what we got when we calculated it the other way too.