MATH 490 Senior Seminar

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A sequence is a list of numbers. But, how do we specify a list of numbers in a meaningful way?

We can:

- List the numbers out
  - Example: $2, 3, 4, 5, 8, 9, 11, 13, 16, ...$
  - These are the prime powers so they are either prime numbers or they are powers of primes
- We can write formulas for the sequences
- We can name the elements (i.e. $a_0, a_1, a_2, ...$)

The best ways to do it are:

1. Closed Form:
   - A function of n like $a_n = 3^n - n^2 + 4$
   - This is ideal for a sequence
2. Recurrence Relations:
   - Example: $a_{n+2} = a_{n+1} + a_n$ where $a_0 = 1$ and $a_1 = 1$
   - Thus, $a_2 = a_1 + a_0 = 2$ and $a_3 = a_2 + a_1 = 3$
3. Generating Functions:
   - These are called "formal power series"
   - The list of numbers are the coefficients
   - We are not concerned about convergence or radii of convergence
   - Example: $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... = \sum_{n=0}^{\infty} a_nx^n$

Advantages and Disadvantages of Generating Functions (GF):

- (+) Access to tools from algebra, calculus, etc.
- (+) Generally compact
- (-) It takes work to get the coefficients; you must evaluate your Taylor series
- (+) We can sometimes find a closed form from it

List of known GF we have already found in other examples:

- $1 + x + x^2 + x^3 + x^4 + ... = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
- $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ... = \sum_{n=0}^{\infty} x^n = \frac{1}{1-ax}$

Today, we will start with a recurrence relation and turn it into a GF. Then we will turn the GF into a closed formula.

Example 1: A sequence defined by $a_n = 2a_n + 1$ where $a_0 = 0$
Write down the first few terms: 0, 1, 3, 7, 15, ... We want to find the GF:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

GOAL 1: We have a trick to turn a recurrence relation into a GF:

- STEP 1: Multiply both sides by x^n so we have

$$a_{n+1} = 2a_n + 1$$

$$a_{n+1}x^n = (2a_n + 1)x^n$$

- STEP 2: Sum for all valid values of n

$$\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty}(2a_n + 1)x^n$$

- STEP 3: Solve for $A(x)$

$$RHS = 2\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} x^n = 2A(x) + \frac{1}{1-x}$$

Divide and multiply by x to make the index and exponent match on the LHS

$$LHS = \frac{1}{x}\sum_{n=0}^{\infty} a_{n+1}x^{n+1}$$

We use a change of variable here to let $m = n + 1$

$$= \frac{1}{x}\sum_{m=1}^{\infty} a_m x^m$$

We lost the first term a_0, so we add and subtract it

$$= \frac{1}{x}(\sum_{m=1}^{\infty} a_m x^m + a_0 - a_0) = \frac{1}{x}(\sum_{m=0}^{\infty} a_m x^m - a_0) = \frac{1}{x}(A(x) - a_0)$$

We have that $a_0 = 0$ and now we can solve for $A(x)$

$$\frac{1}{x}(A(x) - a_0) = 2A(x) + \frac{1}{1-x}$$
$$A(x) - a_0 = 2xA(x) + \frac{x}{1-x}$$
$$(1-2x)A(x) - 0 = \frac{x}{1-x}$$
$$A(x) = \frac{x}{(1-x)(1-2x)}$$

GOAL 2: Turn a GF into a closed formula (we will used known power series and partial fractions

$$\frac{x}{(1-x)(1-2x)} = \frac{P}{1-x} + \frac{Q}{1-2x}$$

$$x = (1-2x)P + (1-x)Q$$

We can try substitution for easy values:
For x = 1:

$$1 = -P$$
$$P = -1$$

For x = $\frac{1}{2}$:

$$\frac{1}{2} = \frac{1}{2}Q$$
$$Q = 1$$

So

$$A(x) = \frac{-1}{1-x} + \frac{1}{1-2x}$$
$$A(x) = \frac{-1}{1-x} + \frac{1}{1-2x}$$
$$A(x) = -\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2^n - 1)x^n$$

So we have that

$$a_n = 2^n - 1$$

There is also a little trick that we can do and obtain another known GF. Consider

$$\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + ...$$

This is just the derivatives:

$$\frac{d}{dx}x + \frac{d}{dx}x^2 + \frac{d}{dx}x^3 + \frac{d}{dx}x^4 + ... = \sum_{n=1}^{\infty} \frac{d}{dx}(x^n) = \frac{d}{dx}(\sum_{n=1}^{\infty} x^n)$$

We know $x^0 = 1$ and that $\frac{d}{dx}1 = 0$, so we see that

$$\frac{d}{dx}(\sum_{n=1}^{\infty} x^n) = \frac{d}{dx}(\sum_{n=0}^{\infty} x^n) = \frac{d}{dx}(\frac{1}{1-x}) = \frac{1}{(1-x)^2}$$

So now we have 3 known generating functions:

- $1 + x + x^2 + x^3 + x^4 + ... = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
- $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ... = \sum_{n=0}^{\infty} x^n = \frac{1}{1-ax}$
- $1 + 2x + 3x^2 + 4x^3 + ... = \sum_{n=1}^{\infty} nx^n = \frac{1}{(1-x)^2}$

Example 2: A sequence defined by $a_{n+1} = 2a_n + n$ where $a_0 = 1$

- Step 1: Multiply by x^n:

$$a_{n+1} = 2a_n + n$$
$$a_{n+1}x^n = (2a_n + n)x^n$$

- Step 2: Summation

$$\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty}(2a_n + n)x^n$$

- Step 3: Solve for A(x)
  - LHS: Make the index and exponent match by multiplying and dividing by x and then adding and subtracting a_0

$$LHS = \frac{1}{x}\sum_{n=0}^{\infty} a_{n+1}x^n = \frac{1}{x}(A(x) - a_0)$$

  - RHS: Separate and then multiply and divide by x where necessary

$$2\sum_{n=0}^{\infty} a_nx^n + x\sum_{n=0}^{\infty} nx^n\frac{1}{x} = 2A(x) + x\sum_{n=0}^{\infty} nx^{n-1}$$

  - We know if $n = 0$, then the first term will vanish and we see:

$$2A(x) + x\sum_{n=0}^{\infty} nx^{n-1} = 2A(x) + x\sum_{n=1}^{\infty} nx^{n-1} = 2A(x) + \frac{x}{(1+x)^2}$$

- All together we see:

$$\frac{1}{x}(A(x) - a_0) = 2A(x) + \frac{x}{(1+x)^2}$$

$$\frac{1}{x}(A(x) - 1) = 2A(x) + \frac{x}{(1+x)^2}$$

$$(1-2x)A(x) = 1 + \frac{x^2}{(1-x)^2}$$

$$A(x) = \frac{1-2x+2x^2}{(1-2x)(1-x)^2}$$

- By partial fractions, we see:

$$\frac{1-2x+2x^2}{(1-2x)(1-x)^2} = \frac{P}{(1-x)^2} + \frac{Q}{(1-x)} + \frac{R}{(1-2x)}$$

  By substituting, we see that $P = -1, Q = 0, R = 2$. So,

$$A(x) = \frac{-1}{(1-x)^2} + \frac{2}{1-2x} = -\sum_{n=1}^{\infty} nx^{n-1} + 2\sum_{n=1}^{\infty} 2^n x^n$$

- To write this generating function as one formula, we can use a change of variable for $m = n - 1$.

$$-\sum_{m=0}^{\infty}(m+1)x^m + \sum_{n=0}^{\infty} 2^{n+1}x^n$$

- Use a change of variables again for $m = n$ and we see:

$$\sum_{n=0}^{\infty}(-n-1+2^{n+1})x^n$$

- Therefore, $a_n = 2^{n+1} - n - 1$

Example 3: Try this one on your own: $a_{n+2} = a_{n+1} + 12a_n$ where $a_0 = 3$ and $a_1 = 5$