Senior Seminar Lecture Notes

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Let's look at the generating function for the Partitions of 'n'.

Let P(n) = number of partitions of 'n'.

Now we can see the following summation:

$$\sum_{n=0}^{\infty} P(x)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x}$$

This product stands for how many of each number we can put into our given partition.

Theorem:

The number of partitions of n into odd parts (odd number of parts with only odd numbers) is equal to the number of partitions of n into distinct parts (no repeating numbers in partition).

$(Ex)n = 7, oddparts:$

$$\begin{gathered} 7 \\ 5+1+1 \\ 3+3+1 \\ 3+1+1+1+1 \\ 1+1+1+1+1+1+1 \end{gathered} \tag{1}$$

So we can see there are 5 ways to make odd partitions in this case

Now for distinct parts:

$n = 7:$

$$\begin{gathered} 7 \\ 6+1 \\ 5+2 \\ 4+3 \\ 4+2+1 \end{gathered} \tag{2}$$

Again, we see there are 5 ways to make partitions, in this case, distinct partitions

Now we are going to prove this using generating functions.

Proof:

We will start with the generating function for the partition into odd parts.

$$O(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i+1}} \tag{3}$$

Now we let D(x) to be the generating function for the partition into n distinct parts.

$$D(x) = (1 + x)(1 + x^2)(1 + x^3)... \tag{4}$$

Each section $(1 + x^n)$ represents if we do or do not we include an 'n'. i.e. 1, 2, 3, etc.

So, we see the following:

$D(x) = \prod_{i=1}^{\infty} 1 - x^i$

$Note: (1 + x^i)(1 - x^i) = (1 - x^{2i})$

$\rightarrow (1 + x^i) = \frac{(1-x^{2i})}{(1-x^i)}$

Now $\quad D(x) = \prod_{i=1}^{\infty} \frac{(1-x^{2i})}{(1-x^i)}$

Let's write out some of the sequence:

$$\frac{1 - x^2}{1 - x} * \frac{1 - x^4}{1 - x^2} * \frac{1 - x^6}{1 - x^3} * \frac{1 - x^8}{1 - x^4} * .... \tag{5}$$

Observing this result, we can see that all of the numerator cancels with all of the even powers in the bottom, thus leaving us with just the odd powers in the denominator.

Thus proving: $\prod_{i=1}^{\infty} \frac{1}{1-x^i} = O(x)$.

Asymptotics

Idea: we can approximate the partition numbers.

Say a_n is asymptotic to a function $f(n)$ if $\lim_{n\to\infty} f(n) = 1$

(written $\rightarrow a_n \sim f(n)$)

This is telling us approximately how fast the function is growing.

(Ex1.)

For the fibonacci numbers we have: $F_n \sim \frac{1}{\sqrt{5}} * (\frac{1+\sqrt{5}}{2})^n$

(Ex2.)

For the catalan numbers we have: $C_n \sim \frac{4^n}{n^{\frac{3}{2}} * \sqrt{\pi}}$

(Ex3.)

For the partition function we have: $P_n \sim \frac{1}{4*\sqrt{3}*n} * e^{\pi * \sqrt{\frac{2n}{3}}}$

Triangulations

Given any n-gon (a shape with 'n' sides), we can always always break up the n-gon into n-2 triangles.

(Ex)

If we look at a rectangle, we can only break it up into two triangles $(4-2=2)$

Question: How many ways to triangulate any given n-gon?

In General:

Let t_n to triangulate an (n+2)-gon into n triangles.

$T(x) = \sum_{i=1}^{\infty}(t_n)x^n$

→ We think about the amount of triangles

→ pick a designated edge, remove the triangle associated with it

→ we are then left with two smaller triangulated shapes.

From this we get $T(x) = xT(x)^2$

*However, we need to account for the "empty" triangulation:

(Ex) a 1-gon, a 2-gon, a 3-gon, all are only 1 triangulation, because we cannot break it up any more from the base shape.

So, this gives us → $T(x) = xT(x)^2 + 1$

*This is the same generating function for the Catalan numbers → $T_n = C_n$

Rooted Plane Trees

Tree:

A graph where there is exactly one way to get between ant vertices.

Rooted Tree:

A graph where one vertex has been chosen (the root) and the tree descends down from the root.

Rooted Plane Tree:

A rooted tree where the order of the branches drawn in the plane matters.

Question:

How many rooted plane trees can we draw given n vertices?

Well, for the n vertices case, we have the following:

$R(x) = \sum_{n=0}^{\infty}(r_n)x^n$, ($r_n$ = the number of rooted plane trees of n-vertices.)

For r_n: we remove the root, which leaves us with a new rooted plane tree from each vertex that was connected to the root.

Idea: the root added with the number of trees we have attached to it is our current number of plane trees, now we can further break down the individual trees:

$R(x) = x(1 + R(x) + R(x)^2 + R(x)^3 + ...)$

*where 1 is for no branches, $R(x)$ is for one branch, $R(x)^2$ is for two branches, and so on...

We can see from $1 + R(x) + R(x)^2 + R(x)^3 + ... = \frac{1}{1-R(x)}$

So now, $R(x) = x * \frac{1}{1-R(x)}$

Now let's find the solution to $R(x)$

$$Solving:$$
$$\begin{gathered}
x = R(x)(1 - R(x)) \\
x = R(x) - R(x)^2 \\
R(x)^2 - R(x) + x = 0 \\
R(x) = \frac{1 \pm \sqrt{1-4x}}{2} \\
R(x) = \frac{1 - \sqrt{1-4x}}{2}
\end{gathered} \tag{6}$$

Looking at this result, we see this is just the Catalan numbers multiplied by X.

So the number of rooted plane trees: $r_n = c_{n-1}$, (b.c $R(x) = xC(x)$.)