Senior Seminar Notes

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Homework Solutions

3. Generating function for $a_k = \binom{n}{k}$
A(x) = $(1+x)^n$
A(1) = $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots$
$\sum_{k=0}^{n} \binom{n}{k}$ = A(1) = $(1+1)^n = 2^n$
a. $\sum_{k=0}^{n}(-1)^n \binom{n}{k}$ = A(-1) = $(1-1)^n = 0^n = 0$
Take k = 3 for an example of an odd k.
1 - 3 + 3 - 1 = 0
When k is odd, numbers appear once positive and once negative to get 0.
Take k = 4 for an example of an even k.
1 - 4 + 6 - 4 + 1 = 0
Let x = -1. Then the generating function becomes $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots$
b. $\sum_{k=0}^{4n} \binom{4n}{4k}$ which is every fourth term.
Let n = 3
$\binom{12}{0} + \binom{12}{4} + \binom{12}{8} + \binom{12}{12}$
A(i) = $\sum_{k=0}^{4n} \binom{4n}{4k} i^k = \binom{4n}{0} + i\binom{4n}{1} - \binom{4n}{2} - i\binom{4n}{3} + \binom{4n}{4} + i\binom{4n}{5} - \binom{4n}{6} - i\binom{4n}{7} + \cdots$
A(-i) = $\sum_{k=0}^{4n} \binom{4n}{4k} (-i)^k = \binom{4n}{0} - i\binom{4n}{1} - \binom{4n}{2} + i\binom{4n}{3} + \binom{4n}{4} - i\binom{4n}{5} - \binom{4n}{6} + i\binom{4n}{7} + \cdots$
So adding A(i) and A(-1) will take away the odd numbers.
A(i) + A(-i) = $2\binom{4n}{0} + 0\binom{4n}{1} - 2\binom{4n}{2} + 0\binom{4n}{3} + 2\binom{4n}{4} + \cdots$
A(1) = $\binom{4n}{0} + \binom{4n}{1} + \cdots$
A(-1) = $\binom{4n}{0} - \binom{4n}{1} + \binom{4n}{2} - \binom{4n}{3} + \cdots$
A(1) + A(-1) = $2\binom{4n}{0} + 0\binom{4n}{1} + 2\binom{4n}{2} + 0\binom{4n}{3} + \cdots$
A(1) + A(-1) + A(i) + A(-i) = $4\binom{4n}{0} + 0\binom{4n}{1} + 0\binom{4n}{2} + 0\binom{4n}{3} + 4\binom{4n}{4} + \cdots = 4\binom{4n}{0} + 4\binom{4n}{4} + \cdots$
$\sum_{k=0}^{4n} \binom{4n}{4k} = \frac{A(1)+A(-1)+A(i)+A(-i)}{4} = \binom{4n}{0} + \binom{4n}{4} + \cdots = \frac{(1+1)^{4n}+(1-1)^{4n}+(1-i)^{4n}+(1+i)^{4n}}{4} = \frac{2^{4n}+0^{4n}+(-4)^{4n}+(-4)^{4n}}{4} = 2^{4n-2} + \frac{1}{2}(-4)^n$

Ring of Formal Power Series
$[[Q]] = \{\sum_{k=0}^{n} a_n x^n\}$
Must be:
Closed under addition and multiplication. Let f(x) = $\sum_{k=0}^{\infty} a_n x^n$ and g(x) =

$\sum_{k=0}^{\infty} b_n x^n$. Then f(x)+g(x) = $\sum_{k=0}^{\infty}(a_n + b_n)x^n$.
f(x)g(x) = $(a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots) = (a_0 + b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots = \sum_{k=0}^{\infty}(x^n(\sum_{i=0}^{\infty} a_ib_{n-i}))$
Associativity and commutativity are inherited from the rationals.
The additive identity is 0(x) = $\sum_{k=0}^{\infty} 0x^n$ and the multiplicative identity is 1(x) = 1 + $\sum_{k=0}^{\infty} 0x^n$.
f(x) has an inverse if there exists a power series g(x) where f(x)g(x) = 1(x) = 1.
Ex: f(x) = $\frac{1}{1-x} = 1 + x + x^2 + \cdots$
f(x)(1-x) - (1-x)$(1 + x + x^2 + \cdots) = 1$
$f^{-1}(x) = 1 - x$

Theorem: The power series f(x) = $a_0 + a_1x + a_2x^2 + \cdots$ has an inverse $f^{-1}(x) = g(x) = b_0 + b_1x + b_2x^2 + \cdots$ if and only if $a_0 \neq 0$.
Proof: If f(x)g(x) = 1 then $a_0b_0 = 1$. So a_0 cannot be 0.
Now suppose $a_0 \neq 0$. We will show f(x) has an inverse by constructing it. b_0 has to be $\frac{1}{a_0}$. Next find b_1. We know $a_0b_1 + a_1b_0 = 0$. So $a_0b_1 + a_1\frac{1}{a_0} = 0$ and therefore $b_1 = -\frac{a_1}{(a_0)^2}$. In general if we have found $b_0, b_1, \cdots, b_{n-1}$ then we can find b_n by solving the equation $\sum_{i=0}^{n} a_ib_{n-i} = 0$.
Ex: (Unfinished)
f(x) = $(1 - x)^n$
$\frac{}{1}1 - x$ is the generating function of all ones.
If n > 0, then f(x) is the generating function for the binomial coefficients. What happens when n<0?
A(x) = $\frac{1}{(1-x)^n} = (\frac{1}{1-x})^n = (1+x+x^2+\cdots)(1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots$ = 1 + nx + ?
The coefficient on x^k is all of the ways we can choose n integers that sum to k.
Counting solutions to $e_1 + e_2 + \cdots + e_n$ = k where e_i are non-negative integers that are the exponents of x in each of the power series.
How many ways can we write k as the sum of n non-negative integers?
Let us take k balls and put them in n boxes. Draw a vertical line to divide one box from the next: o l oo l l o
We need n-1 vertical lines and k o's.
Every possible such configuration of writing down n-1 vertical lines and k o's gives us a way of putting k things in n boxes.