Math 490 - Fall 2015
Homework 3
Due March 16, 2017
Algebra With Personality=Combinatorics.

—Dave Bayer

Turn in:

(1) Let a_n be the number of orders in which n people can finish a race if ties are allowed. (The first few terms for $n \geq 0$ are: 1, 1, 3, 13, 75, ...)
  (a) Find the exponential generating function
  $$g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$
  Hint: first consider all the ways to break the n people into groups that "tie together" and then combine them in the same way we did for permutations.
  (b) Check that your function works in Mathematica (or equivalent) and use it to find the next 2 terms in the sequence.

(2) Recall that the exponential generating function for the number of cycles of length n was $C(x) = \ln\left(\frac{1}{1-x}\right)$.
  (a) Find the exponential generating function $E(x)$ for the number of cycles of even length ($E(x) = \frac{x^2}{2} + \frac{x^4}{4} + \cdots$) and the exponential generating function $F(x)$ for the number of cycles of odd length ($F(x) = \frac{x}{1} + \frac{x^3}{3} + \cdots$)
  (b) Use this to find the exponential generating functions for the number of permutations consisting only of cycles of even length, and the number of permutations consisting only of cycles of odd length.

(3) Let D_n denote the number of derangements of n. (Permutations without fixed points.) Show that
$$D_n = nD_{n-1} + (-1)^n$$
by using the exponential generating function for derangements, $D(x) = \frac{e^{-x}}{1-x}$.