Math 490 - Fall 2015
Homework 3
Due March 2, 2017

In my opinion a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy—an opinion, moreover, which has been expressed by many philosophers.

—Henry Lebesgue

Turn in:

(1) Use Stirling's approximation (look it up!) to prove that the Catalan numbers are asymptotic to $\frac{4^n}{n^{3/2}\sqrt{\pi}}$, i.e. prove that

$$\lim_{n\to\infty} \frac{\left(\frac{4^n}{n^{3/2}\sqrt{\pi}}\right)}{C_n} = 1.$$

(2) Simplify

$$f(x) = \prod_{i=0}^{\infty}(1 + x^{2^i})$$

by computing $(1-x)\cdot f(x)$, (Hint: try multiplying out $(1-x)\cdot\prod_{i=0}^{n}(1+x^{2^i})$ first) and give a combinatorial explanation of the result.

(3) Suppose that we want to count paths like Dyck paths where instead of just steps up and down, we are also allowed to take horizontal steps, however the path is still not allowed to cross below the x-axis, and must end at height 0. (Note that in this case a path doesn't have to have an even length!)

(a) Let M_n denote the number of such paths of length n. By drawing all of the paths, find M_n for $n \le 4$. (Hint: $M_0 + M_1 + M_2 + M_3 + M_4 = 17$.)

(b) Find a closed form for the generating function for these numbers

$$M(x) = \sum_{n=0}^{\infty} M_n x^n.$$

(Hint: Use the same trick we used to find the generating function for Dyck paths.)

(c) Use a software package (Mathematica, Sage, Wolfram Alpha, etc) to find the first 10 terms of this sequence by computing the taylor series of the function you found in part b. Do they agree with the numbers you found in part a?

(4) Look on the OEIS and pick a sequence we haven't talked about in class that looks interesting, and has a generating function listed. (Under formula look for G.F.) Give the reference number A?????? and describe the sequence in a few sentences.