Math 490 - Fall 2015
Homework 1
Due Febuary 21, 2017

From this proposition it will follow, when arithmetical addition has been defined, that $1+1=2$. *The above proposition is occasionally useful.*

— Alfred North Whitehead and Bertrand Russell on page 379 of Principia Mathematica

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Turn in:

(1) (a) Find a closed form for the generating function for the sequence $a_0 = 0$, $a_n = \frac{1}{n}$, $n \geq 1$.
(b) The nth harmonic number is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Find a closed form for the generating function for the harmonic numbers.

(2) Let $F(x)$ be the generating function for the sequence $a_0 = 1$, $a_{2n} = a_n + a_{n-1}$, $a_{2n+1} = a_n$.
(a) Write out the first ten terms of this sequence.
(b) Show that
$$F(x) = (1 + x + x^2)F(x^2).$$

(3) Consider the Perrin sequence, which we can define recursively as follows: set $r_0 = 3, r_1 = 0, r_2 = 2$, and for all $n \geq 3$ define $r_n = r_{n-2} + r_{n-3}$.

Let $R(x)$ denote the generating function $R(x) = \sum_{n=0}^{\infty} r_n x^n$ for the Perrin sequence.

(a) Prove that $R(x) = \frac{3-x^2}{1-x^2-x^3}$.
(b) Prove that $r_n = r_{n-1} + r_{n-5}$, for all $n \geq 5$.

(4) Here's a fun/strange fact: if p is a prime number, then p divides r_p, the p-th Perrin number. We prove this in this problem, using the techniques of generating functions, as follows:

(a) Define $Q(x) = R(x) - 3$; that is, $Q(x)$ is the same power series as $R(x) = \sum_{n=0}^{\infty} r_n x^n$, except we make the constant term zero for calculational convenience.
Show that
$$Q(x) = (-x) \cdot \frac{d}{dx}\left(\ln(1 - x^2 - x^3)\right).$$
(b) Show that if $Q(x)$ is any power series $\sum_{n=0}^{\infty} r_n x^n$, then
$$\left.\frac{d^p}{dx^p}\left(Q(x)\right)\right|_{x=0} = p! \cdot r_p.$$
(c) By using the product rule on the equation from part a, show that
$$\left.\frac{d^p}{dx^p}\left(Q(x)\right)\right|_{x=0} = (-p) \cdot (p!) \cdot (\text{ the coefficient of } x^p \text{ in the Taylor series for } \ln(1 - x^2 - x^3)).$$
(d) Using the above, prove that p is a factor of r_p whenever p is prime.
(Hint: the coefficient of x^p in the Taylor series for $\ln(1 - x^2 - x^3)$ may be a fraction, however use the generating function from problem 1.a to see that the coefficient on x^p is a sum of fractions whose denominators are all less than p, and so this fraction can't have a p in the denominator.)