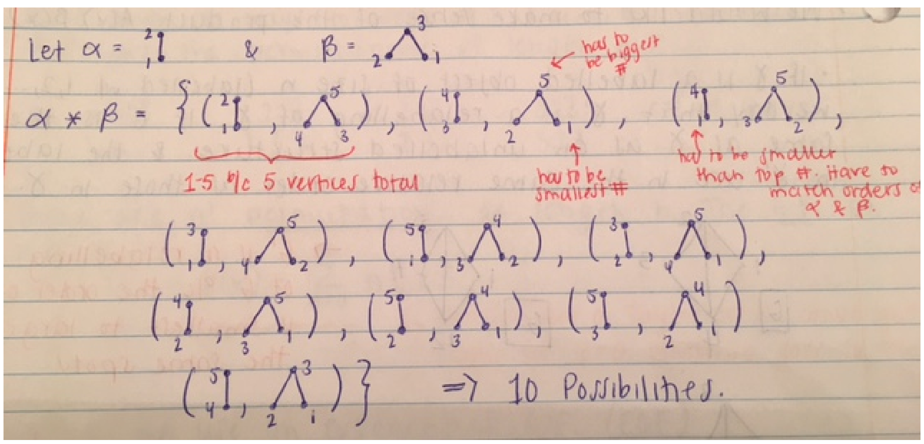
March 3, 2017

1 Product of Two Labeled Structures

α is a labeled structure of size l.
β is a labeled structure of size k.

(α * β) = {(α', β') | α' and β' are labeled with the numbers 1,2,...,l+k, and α' is a relabeling of α, and β' is a relabeling of β.}

2 Labeled Rooted Plane Trees



Question: If α has size k and β has size l, how many elements are in (α * β)?

Answer: There are $\binom{k+l}{k}$ elements in (α * β).

Say that A is a collection of labeled objects with exponential generating function f(x), and B is another collection with exponential generating function g(x).

Let $\bigcup_{\alpha\in A,\beta\in B}$ [α*β].

Find the exponential generating function h(x) for C.

Let h(x) = $\sum_{n=0}^{\infty} \frac{c_n}{n!} x^n$

Let f(x) $= \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$

Let g(x) $= \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$

Every object in C of size n is constructed by taking an α of size k from A and a β from B of size (n-k). This choice of α and β gives us $\binom{n}{k}$ objects in C.

$C_n = \sum_{k=0}^{n} a_k \; b_{n-k} \binom{n}{k}$, the number of objects in C.

a_k is the possible α's, b_{n-k} is the possible β's, $\binom{n}{k}$ is the total number of things.]

Now, h(x) $= \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n$

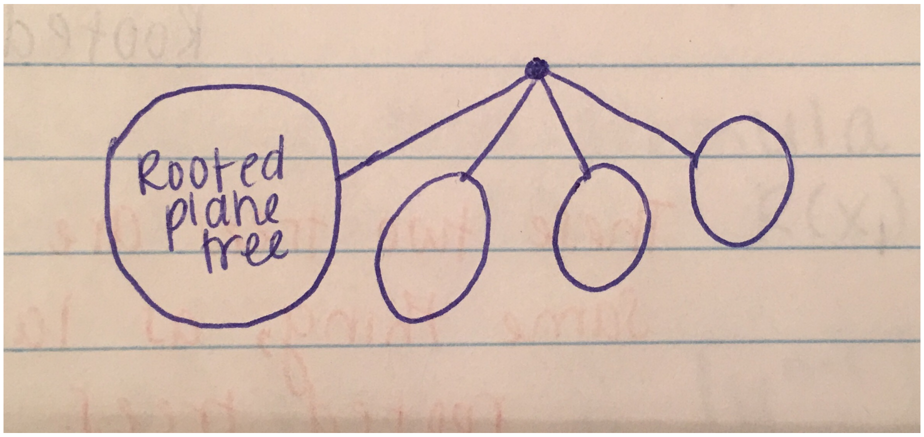
$= \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_k \; b_{n-k} \binom{n}{k}) \frac{x^n}{n!}$

$= \sum_{n=0}^{\infty} (\sum_{k=0}^{n} (\frac{a_k}{k!})(\frac{b_{n-k}}{(n-k)!}) \; \frac{n! x^n}{n!}$

$=(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n)(\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n)$

So, h(x) = f(x)g(x).

2.1 Count Labeled Rooted Plane Trees



Recall: (Unlabeled) Rooted Plane Trees

- Start with a root
- Any rooted plane tree consists of a root along with a sequence of rooted plane trees that get hung from the root in order.
- So, p(x) $= x(1 + p(x) + p(x)^2 + p(x)^3 + ...) = x(\frac{1}{1-p(x)})$

This same construction works for the exponential generating function for labeled rooted plane trees. But same function for both.

Question: How many labeled rooted plane trees are there?

$p(x)(1 - p(x)) = x$

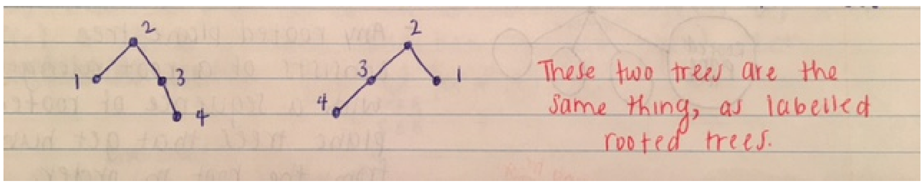
$p(x) - p(x)^2 = x$

$0 = p(x)^2 - p(x) + x$

$p(x) = \sum_{n=1}^{\infty} c_{n-1}x^n$

Answer: $n![x^n]p(x) = n!\ c_{n-1} = \frac{n!}{n}\binom{2(n-1)}{(n-1)}) = \frac{(2n-2)!}{(n-1)!}$

Question: What if we remove the "plane" requirement? (Labeled Rooted Trees)



- Construction for labeled rooted trees is similar.
  - Start with a root.
  - Below it attach an unordered set of labeled rooted plane trees.

Let R(x) = exponential generating function of labeled rooted trees.

So, R(x) $= x(1 + R(x) + \frac{R(x)^2}{2!} + \frac{R(x)^3}{3!} + ... + \frac{R(x)^n}{n!} = xe^{R(x)}$

(The n! accounts for swapping n number of trees that are the same.)

2.2 Lagrange Inversion Formula

Suppose f(x) and g(x) are formal power series and g(0)=1. Suppose f(x) = x(g(f(x))).

Then,

$[x^n]f(x) = \frac{1}{n}[u^{n-1}](g(u))^n$.

Let's apply this formula to labeled rooted trees. In this case, F(x)=R(x) and G(x)=e^x.

So,

$x^n R(x) = \frac{1}{n}[u^{n-1}](e^u)^n = \frac{1}{n}[u^{n-1}]e^{nu} = \frac{1}{n}(\frac{n^{n-1}}{(n-1)!}) = \frac{n^{n-1}}{n!}$

R(x) $= \sum_{n=0}^{\infty} \frac{n^{n-1}}{n!} x^n$

So, there are n^{n-1} labeled rooted trees of size n.