Introduction/Review

In the previous lecture, we discussed how, given a recurrence relation, to find the generating function and closed formula of a sequence. This is done using the following algorithm:

- Recurrence Relation to Generating Function
  1. Multiply both sides of the recurrence relation by x^n
  2. Sum both sides for all valid values of n
  3. Solve for A(x), the generating function
- Generating Function to Closed Formula
  1. Use the method of partial fractions to find the fraction decomposition of A(x)
  2. Write A(x) as a sum of known power series
  3. The closed form, a_n, of the sequence is the coefficient of x^n in the constructed summation

Fibonacci Sequence

Given the recurrence relation,

$$a_{n+2} = a_{n+1} + a_n;\ a_n = n$$

find the generating function and closed formula of the Fibonacci sequence.

Recurrence Relation to Generating Function

Using the previously stated algorithm, we know

$$\sum_{n=0}^{\infty} a_{n+2}x^n = \sum_{n=0}^{\infty}(a_{n+1} + a_n)x^n = \sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n$$

Left Hand Side

Using $m = n + 2$:

$$\sum_{m=2}^{\infty} a_m x^{m-2}$$
$$= \frac{1}{x^2} \sum_{m=2}^{\infty} a_m x^m$$
$$= \frac{1}{x^2} [\sum_{m=0}^{\infty} a_m x^m - 0 - x^1]$$
$$= \frac{1}{x^2} [\sum_{m=0}^{\infty} a_m x^m - x]$$
$$= \frac{1}{x^2} [A(x) - x]$$

Right Hand Side

We know that $\sum_{n=0}^{\infty} a_n x^n = A(x)$.

Using $m = n + 1$:

$$\sum_{m=1}^{\infty} a_m x^{m-1} + A(x)$$
$$= \frac{1}{x} \sum_{m=1}^{\infty} a_m x^m + A(x)$$
$$= \frac{1}{x} [\sum_{m=0}^{\infty} a_m x^m - 0] + A(x)$$
$$= \frac{1}{x} A(x) + A(x)$$

Solve for A(x)

$$\frac{1}{x^2} [A(x) - x] = \frac{1}{x} A(x) + A(x)$$

$$A(x) - x = xA(x) + x^2 A(x)$$

$$A(x)(1 - x - x^2) = x$$

$\boxed{\boldsymbol{A(x) = \frac{x}{1 - x - x^2}}}$ is the generating function for the Fibonacci sequence.

Generating Function to Closed Formula

Find the fraction decomposition of A(x) using the method of partial fractions, and write the denominator in the form $(1-\alpha x)(1+\beta x)$.

$$A(x) = \frac{x}{(1-\alpha x)(1+\beta x)} = \frac{P}{(1-\alpha x)} + \frac{Q}{(1+\beta x)}$$

$$x = P(1+\beta x) + Q(1-\alpha x)$$

Using $x = \dfrac{1}{\alpha} = \beta$,

$$\begin{aligned}\beta &= P(1+\beta^2)\\ &= P(1+1-\beta) \text{ because } \beta^2 = 1-\beta\\ &= P(2-\beta)\end{aligned}$$

$$\boxed{P = \frac{\beta}{2-\beta}}$$

Using $x = \dfrac{-1}{\beta} = -\alpha$,

$$\begin{aligned}-\alpha &= Q(1+\alpha^2)\\ &= Q(1+1+\alpha) \text{ because } \alpha^2 = 1+\alpha\\ &= Q(2+\alpha)\end{aligned}$$

$$\boxed{Q = \frac{-\alpha}{2+\alpha}}$$

Find α and β using

$$(1-\alpha x)(1+\beta x) = 1 - x - x^2 = -(x-x_1)(x-x_2). \qquad (*)$$

Apply Quadratic Formula to $(*)$:

$$x = \frac{-1\pm\sqrt{5}}{2}, \text{ so } x_1 = \frac{-(1+\sqrt{5})}{2} \text{ and } x_2 = \frac{-1+\sqrt{5}}{2}$$

$$\begin{aligned}(*) &= -(x + \frac{1+\sqrt{5}}{2})(x - \frac{-1+\sqrt{5}}{2})\\ &= (\frac{1+\sqrt{5}}{2} + x)(\frac{\sqrt{5}-1}{2} - x)\\ &= (1 - x/\frac{\sqrt{5}-1}{2})[1 + (\frac{\sqrt{5}-1}{2})x]\\ &= (1 - (\frac{\sqrt{5}+1}{2})x)(1 + (\frac{\sqrt{5}-1}{2})x),\end{aligned}$$

where $\boldsymbol{\dfrac{\sqrt{5}+1}{2} = \alpha}$ and $\boldsymbol{\dfrac{\sqrt{5}-1}{2} = \beta}$.

Substitute values for α and β into P and Q:

$P = \frac{\beta}{2-\beta}$ $\qquad$ $Q = \frac{-\alpha}{2+\alpha}$

$2-\beta = 2 - \frac{\sqrt{5}-1}{2} = \sqrt{5}\beta$ $\qquad$ $2+\alpha = 2 + \frac{\sqrt{5}+1}{2} = \sqrt{5}\alpha$

$\boldsymbol{P = \frac{\beta}{\sqrt{5}\beta} = \frac{1}{\sqrt{5}}}$ $\qquad$ $\boldsymbol{Q = \frac{-\alpha}{\sqrt{5}\alpha} = \frac{-1}{\sqrt{5}}}$

Write A(x) as a sum of known power series to find the closed formula, a_n:

$$\begin{aligned} A(x) = \frac{P}{(1-\alpha x)} + \frac{Q}{(1+\beta x)} &= \frac{1/\sqrt{5}}{(1-\alpha x)} + \frac{-1/\sqrt{5}}{(1+\beta x)} \\ &= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\alpha^n x^n - \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}(-\beta)^n x^n \\ &= \sum_{n=0}^{\infty}\left(\frac{\alpha^n}{\sqrt{5}} - \frac{(-\beta)^n}{\sqrt{5}}\right)x^n \end{aligned}$$

so $a_n = \frac{1}{\sqrt{5}}(\alpha^n - (-\beta)^n)$.

$\boxed{\boldsymbol{a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}}$ is the closed formula for the Fibonacci sequence.

The Ring of Formal Power Series

Definition: For every sequence $(a_n)_{n\geq 0}$ with $a_n \in \mathbb{Q}$ (or $\mathbb{R}$), define

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

We denote this set $\mathbb{Q}[[x]]$.

For two power series, we define

$$\begin{aligned} A(x) + B(x) &= \sum_{n\geq 0}(a_n + b_n)x^n \\ A(x)\cdot B(x) &= \sum_{n\geq 0} C_n x^n, \end{aligned}$$

where

$$C_n = \sum_{\substack{i+j=n \\ i,j\geq 0}} a_i b_j = \sum_{j=0}^{n} a_j b_{n-j} = \sum_{j=0}^{n} a_{n-j} b_j.$$

We say that $\sum_{n\geq 0} b_n x^n$ is the **multiplicative inverse** $(\sum_{n\geq 0} a_n x^n)^{-1}$ if

$$(\sum_{n\geq 0} a_n x^n)(\sum_{n\geq 0} b_n x^n) = 1 + 0x + 0x^2 + \ldots = 1.$$