Senior Seminar Notes

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1 Homework Solution

Perrin Sequence: $a_0 = 3, a_1 = 0, a_2 = 2, a_3 = 3, a_4 = 2, a_5 = 5, a_6 = 5, a_7 = 7, a_8 = 10, a_9 = 12, \ldots, a_p$. a_p is always divisble by p. If n divides a_n, but n is not prime, then n is called a Perrin Pseodoprimel smallest example is 600,000.

(a) Define Q(x)=R(x)-3; that is, Q(x) is the same power series as $R(x) = \sum_n^\infty r_n x^n$. Show that $Q(x) = (-x) * \frac{d}{dx} * (ln(1 - x^2 - x^3)$.

$R(x) = \sum_{n=0}^\infty a_n x^n$ is the generating function for Perrin Sequence.

$Q(x) = R(x) - 3$

$Q(x) = \frac{3-x62}{1-x^2-x^3} - 3 = \frac{2x^2+3x^3}{1-x^2-x^3}$

$-x\frac{d}{dx}((Ln(1 - x^2 - x^3)) = \frac{2x^2+3x^3}{1-x^2-x^3}$

As you can see $Q(x) = -x\frac{d}{dx}((Ln(1 - x^2 - x^3))$

(b) Show that if $Q(x)$ is any power series $\sum_{n=0}^\infty r_n x^n$, then $\frac{d^p}{dx^p}[Q(x)]_x = p^1 r_p$.

$Q(x) = \sum_{n=0}^\infty r_n x^n = r_0 + r_1 x + r_2 x^2 + \ldots + r_p x^p + r(_p+1)x^(p+1) + \ldots$ taking the p derivative of the polynomial less than p=0 and plugging in x=0, we end up with $p!r_p$

(c) By using the product rule on the equation from part a, show that $\frac{d^p}{dx^p}[Q(x)]_x = (-p) * (p!)*$ (the coefficient of x^p in the Taylor Series for $Ln(1 - x^2 - x^3)$).

$\frac{d^p}{dx^p}(Q(x)) = \frac{d^p}{dx^p}[(-x)\frac{d}{dx}]Ln(1 - x^2 - x^3)$

$= \frac{d^(p-1)}{dx_(p-1)}[(-1)\frac{d}{dx}Ln(1 - x^2 - x^3) + (-x)\frac{d^2}{dx^2}[Ln(1 - x^2 - x^3)]$

If we keep taking p derivatives we get:

$(-p_1)\frac{d^p}{dx_p)}Ln(1 - x^2 - x^3) + (-x) + \frac{d^(p+1)}{dx_(p+1)}(1 - x^2 - x^3)$, x=0.

(d) Using the above, p rove that p is a factor of r_p whenever p is prime.

We know that Ln(1-y)=-$\sum_{n=1}^\infty \frac{y^n}{n}$.

Let $y = x^2 - x^3$, then

$Ln(1 - x^2 - x^3) = -\sum_{n=1}^\infty \frac{(x^2-x^3)^n}{n} = \frac{x^2+x^3}{1} + (x^2 + x^3)^2 2 + \frac{(x^2+x^3)}{3} + \ldots + \frac{(x^2+x^3)^p}{p}$

Coefficient on x^p comes only form terms with denominators less than p in this sum. So r_p is a sum of fractions whise denominations are all less than p. So the denominator of r_p is not divisble by p.

$p!a_p = -p-!r_p$

where $p!a_p$ is only divisble by p one time and $-pp!r_p$, where p divides this side out at least 2 times.

2 Labelled Structures

When we have an object of size n, we are going to five its components labels from 1 up to n. We call the structures different if they are labelled differently.

2.1 Labelled Graphs

Graph with n vertices, label the vertices $1, 2, \ldots, n$.

2.2 Labelled Structures

Write down all the ways to write down numbers $1, 2, 3, \ldots, n$ in a line. These are the permutations of length n.

Example: Say n=3, there are 6 permuations.

So there are n! permutations of length n.

The ordinary generating function for the permutation is $p(x) = \sum_{n=0}^{\infty} n! x^n$. This sum as a taylor series converges only at x=0.

Instead, we use an exponential generating function.

3 Exponential Generating Function

We say that the exponential generating function (EGF) of a sequence $a_0, a_1, a_2, \ldots, a_n$ is $A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$.

The EGF will be much more useful for counting labelled structures. The EFG for the permutations is $Q(x) = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

If we're counting some set of objects that are labelled and are the disjoint union of two sets with EGFs A(x) and B(x), then the exponential generating funtion for this set is A(x)+B(x).

We would like to make sense of the product A(x)B(x).

If γ is a labelled object of size n (labelled with $1, 2, \ldots, n$), we say that γ^1 is a relabelling of γ.

If γ^1 is the same as γ as an unlablled structure and the labels in γ^1 are the same relative order as those in γ.

Product of Two Labelled Structures:

α is a labelled structure if size l.

β is a labelled structure of size k.

$(\alpha)(\beta) = (\alpha^1, \beta^1) | \alpha^1 \text{and} \beta^1 \text{and labelled with the numbers} (1, 2, \ldots, l+k) \text{and} \alpha^1 \text{is a relabelling of} \alpha \text{and} \beta^1 \text{is a relabel}$