Math 490 - Spring 2017
Lecture Notes 2/21/17
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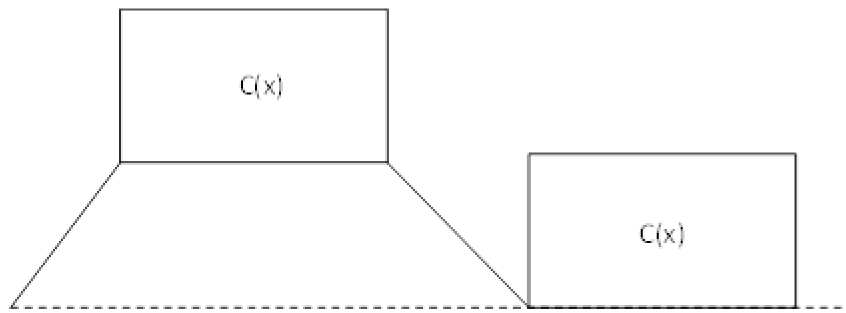
Dyck Paths: Paths with up and down steps that start at zero and never cross below the x-axis.

Any Dyck Path can be decomposed as:



Generating function: $C(x) = x \cdot C(x)^2 + 1$ Need to find a way to solve this.

$x \cdot C(x)^2 - C(x) + 1 = 0$ Solve for $C(x)$ using quadratic formula.

$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$ The $\pm$ is ambiguous, which one do we want?

Consider $\sqrt{1-4x} = (1-4x)^{1/2}$. Using the generalized binomial theorem:

$$(1-4x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n}(-4x)^n = 1 + \binom{1/2}{1}(-4x) + ...$$

Putting this series in place of $\sqrt{1-4x}$: $C(x) = \frac{1+\sqrt{1-4x}}{2x} = \frac{1}{x}$

This result does not make sense for generating functions, which count things.

Therefore, we want $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, which does not result in negative powers of x.

Next, can we write the closed form of $C(x) = \frac{1-\sqrt{1-4x}}{2x}$?

Closed Form for nth Coefficient

$[x^n]C(x) = [x^n]\frac{1-\sqrt{1-4x}}{2x}$, however we want to consider $[x^n]\frac{-\sqrt{1-4x}}{2x}$ and assume n> 0:

$$[x^n]\frac{-\sqrt{1-4x}}{2x} = [x^{n+1}]\frac{-\sqrt{1-4x}}{2} = -\frac{1}{2}[x^{n+1}](1-4x)^{1/2} = -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1}$$

Simplifying:

$$\binom{1/2}{n+1} = \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\cdot\ldots\cdot(\frac{1}{2}-n)}{(n+1)!} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdot\ldots\cdot(-\frac{2n-1}{2})}{(n+1)!} = \frac{(-1)^n}{2^{n+1}}\frac{1\cdot1\cdot3\cdot5\cdot7\cdot\ldots\cdot(2n-1)}{(n+1)!}$$

$$= \frac{(-1)^n}{2^{n+1}}\frac{(2n-1)!}{(n+1)!\cdot2\cdot4\cdot6\cdot\ldots\cdot(2n-2)} = \frac{(-1)^n}{2^{n+1}}\frac{(2n-1)!}{(n+1)!(n-1)!(2^{n-1})} = \frac{(-1)^n(2n-1)!}{4^n(n+1)!(n-1)!}$$

$$(-4)^{n+1} = (-1)^{n+1}\cdot(4)^{n+1}$$

Then:

$$[x^n]C(x) = -\frac{1}{2}\frac{(-1)^n(2n-1)!}{4^n(n+1)!(n-1)!}\cdot(-1)^{n+1}\cdot(4)^{n+1} = \frac{2(2n-1)!}{(n+1)!(n-1)!}$$

Multiplying by $\frac{n}{n}$:

$$[x^n]C(x) = \frac{2n(2n-1)!}{(n+1)!(n-1)!n} = \frac{(2n)!}{(n+1)!n!} = \frac{1}{(n+1)}\frac{(2n)!}{n!\cdot n!} = \frac{1}{(n+1)}\cdot\binom{2n}{n}$$

The result is called the Catalan number and represents the number of Dyck paths there are that have length $2n$.

Recall the Catalan numbers: $1, 1, 2, 5, 14, \ldots$

Corollary: $\binom{2n}{n}$ is divisible by $(n+1)$.

Partitions

A partition of the integer n is any way of writing n as a sum of positive integers where order in the sum does not matter.

Partitions of 4:

$4 = 1+1+1+1 = 2+2 = 2+1+1 = 3+1 = 4$
There are five partitions for 4.

Partitions (contd)

Let $P(n)$ = number of partions of n. For example, $P(4) = 5$.

To "draw" partitions we use Ferrers diagrams. Write each term in a partition as a row of dots. For example:

$4 = 2 + 1 + 1$ can be represented as:

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•
•

Let $P_k(n)$ = number of partitions of n into exactly k pieces. For example,

$P_2(4) = 2$ and $P_3(4) = 1$

Note that $P(n) = \sum_{k=1}^{n} P_k(n)$

<u>Observations</u>

If we take a partition of n and if its smallest term is a 1 we can remove that 1 and get a partition of $(n-1)$.

If we have a partition that does not have any 1's then we can subtract 1 from every term in the partition.

If the partition has k parts we get a partition of $(n-k)$ into k parts.

$P_{k-1}(n-1)$ is how we make all partitions of n with k parts that end with a 1.

$P_k(n-k)$ is how we make all partitions of n into k parts where every term is bigger than 1.

Using $P_k(n) = P_{k-1}(n-1) + P_k(n-k)$ along with $P_n(n) = 1$ and $P_k(n) = 0$ provided $k > n$:

n	$P(n)$	$P_1(n)$	$P_2(n)$	$P_3(n)$	$P_4(n)$	$P_5(n)$	$P_6(n)$
1	1	1	0	0	0	0	0
2	2	1	1	0	0	0	0
3	3	1	1	1	0	0	0
4	5	1	2	1	1	0	0
5	7	1	2	2	1	1	0
6	11	1	3	3	2	1	1

Note: There is no known closed form for $P(n)$.

Generating Function for $P(n)$

$P(x) = \sum_{n=0}^{\infty} P(n)x^n$

$\frac{1}{1-x}$ counts ways to write n as a sum of just 1's.

$\frac{1}{1-x^2}$ counts ways to write n as a sum of just 2's.

$\frac{1}{1-x^k}$ counts ways to write n as a sum of just k's.

Multiplying all of these together:

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = P(x)$$