Math 490 - Spring 2017
Class Notes - 02/16/2017

How many ways can we write the number n as the sum of k nonnegative integers?

$e_1 + e_2 + e_3 + \cdots + e_k = n$

Same as asking, "how many ways can we put n objects into k boxes in a row?"
We can represent this by drawing circles for the objects and vertical lines to divide the boxes

So, we need n circles and $k - 1$ vertical lines.

So... how many ways can we write down a string of n circles and $k - 1$ lines?

ex. OO | OO | | O represents 2 objects in box 1, 2 in box 2, 0 in box 3, and 1 in box 4

Then we have $n + k - 1$ symbols to write down. Thus, there are $\binom{n+k-1}{n}$ ways to choose n of these symbols to be O's and the rest |'s.

$\Rightarrow$ There are $\binom{n+k-1}{n}$ ways to write n as a sum of k non-negative integers.

$$\therefore \frac{1}{(1-x)^k} = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right)\cdots$$
$$= \sum_{n=0}^{\infty} \binom{n+k-1}{n} x$$

Generalized Binomial Theorem

Let $\alpha \in \mathbb{Q}$. Then, $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, where the *generalized binomial coefficient* $\binom{\alpha}{n} = \frac{(\alpha)(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}$.

Multiplying generating functions $[x^n](A(x)B(x))$ counts how many ways we can create an object of size n by combining objects of size a from $A(x)$ and objects of size b from $B(x)$ where $a + b = n$.

example. How many ways can you make change for a dollar using pennies, nickels, dimes, and quarters? We can answer this using generating functions.

$\frac{1}{1-x} = 1 + x + x^2 + \dots$ counts how many ways we can get n cents using only pennies.

$\frac{1}{1-x^5} = 1 + x^5 + x^{10} + x^{15} + \ldots$ counts how many ways we can get n cents using only nickels.

Similarly, $\frac{1}{1-x^{10}}$ counts how many ways to get n cents using dimes and $\frac{1}{1-x^{25}}$ counts how many ways to get n cents using only quarters.

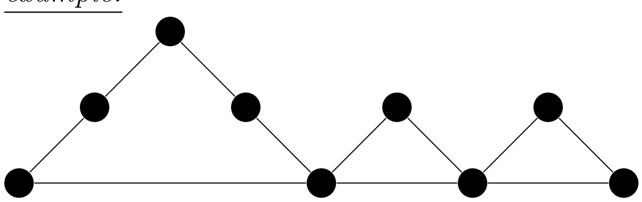
$\Rightarrow C(x) = (\frac{1}{1-x})(\frac{1}{1-x^5})(\frac{1}{1-x^{10}})(\frac{1}{1-x^{25}})$ is the generating function we can use to count the number of ways to make change for n cents.

$\Rightarrow [x^{100}]C(x)$ is the number of ways we can count change for a dollar using pennies, nickels, dimes, and quarters.

Dyck Paths

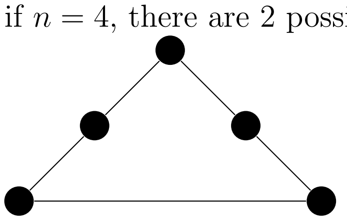
- Walk of length $2n$ where, at every step, we either go up or down.
- Always stay above or on the x-axis
- Start and end at the x-axis

example.

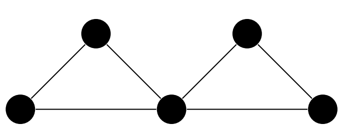


Dyck Path of length 8

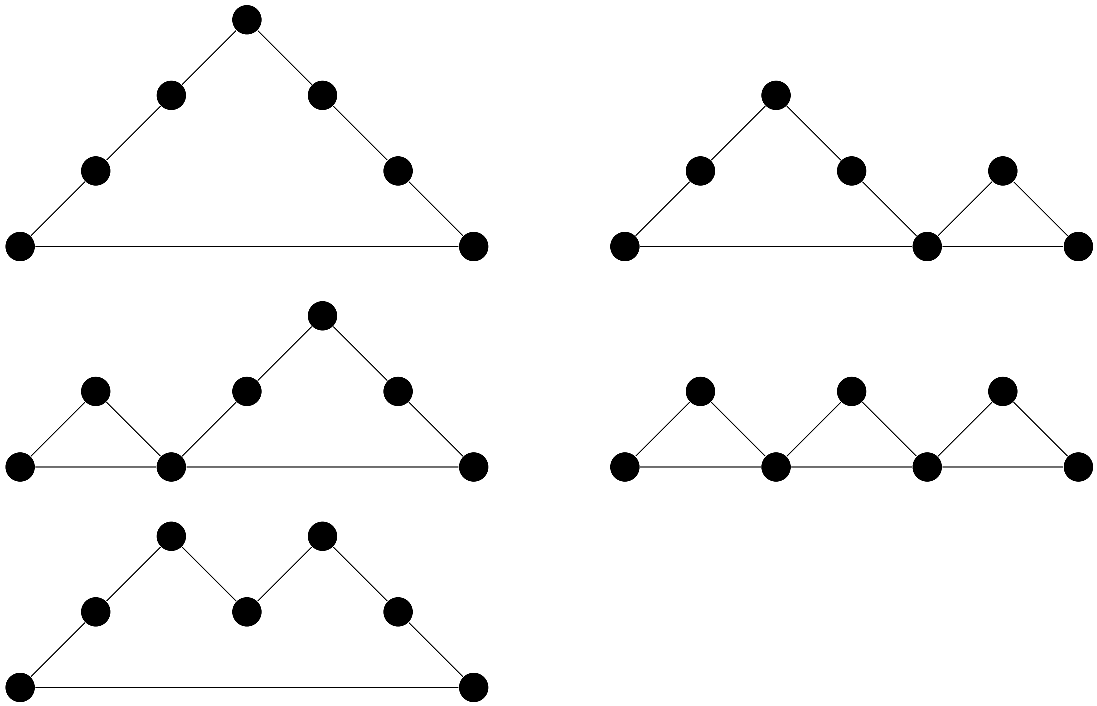
How many Dyck Paths are there of length n?
if $n = 0$, there is one possibility (empty path)
if $n = 2$, there is one possibility (up then down)
if $n = 4$, there are 2 possibilities



or

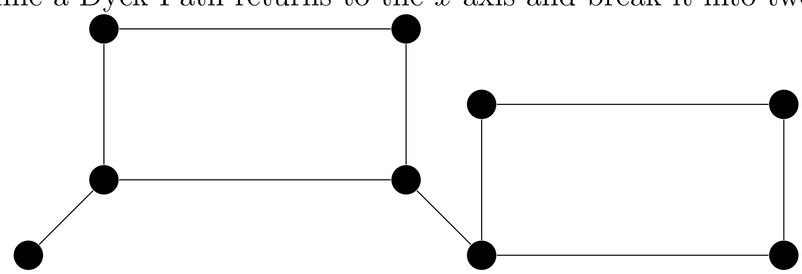


if $n = 6$, there are 5 possibilities



So, our sequence so far is 1, 1, 2, 5, . . . , and we want to find a generating function.

Our solution is to break a Dyck Path into smaller Dyck Paths. We can always take the first time a Dyck Path returns to the x-axis and break it into two paths there, as shown below:



where the first box represents any Dyck Path of length k, the second box represents any Dyck path of length l, and so the whole Dyck Path has length $k + l + 2$.

Let $C(x)$ be the generating function for the count of Dyck Paths of length $2n$.

$\Rightarrow C(x) = 1 + x + 2x^2 + 5x^3 + \dots$

From decomposing any Dyck Path the way we did earlier, we have

$C(x) = xC(x)C(x) + 1$ where x corresponds to the up step and down step, $[C(x)]^2$ corresponds to the two smaller Dyck Paths, and 1 accounts for the empty path.

Then, $C(x) = xC(x)^2 + 1$

Solving for $C(x)$, we have

$$\begin{aligned} 0 &= xC(x)^2 - C(x) + 1 \\ C(x) &= \frac{1 \pm \sqrt{1-4x}}{2x} \end{aligned}$$