**Math 465 - Spring 2021** February 4th, 2021
**Group Discussion 2**

*I'm a mathematical optimist: I deal only with positive integers.*

— Tendai Chitewere

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(1) Let $a, b$ be positive integers and $d = \gcd(a, b)$. Prove that $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.
(2) Prove that if $p$ is a prime number and $p \mid ab$ then $p \mid a$ or $p \mid b$. Then give a counterexample to this statement if $p$ is not a prime number.

We will use this to prove the following (important) theorem.

> **Fundamental Theorem of Arithmetic:** (Unique Factorization) For every integer $n > 1$ there exist unique primes $p_1 < p_2 < \cdots p_k$ and positive integers $a_1, a_2, \ldots, a_k$ such that
> $$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}.$$

(3) Let $n \in \mathbb{Z}$ with $n > 1$.
  (a) Prove, by induction that there exist the primes $p_1, p_2, \ldots$ and integers $a_1, a_2, \ldots$ satisfying the equation above.
  (b) Prove, by contradiction, that these primes and integers are unique. (Hint: suppose there exist integers with different factorizations, and let $m$ be the smallest number with distinct factorizations... This is where you may want to use problem 2 as well.)
(4) Suppose $a, b \in \mathbb{Z}$, not both zero. Define $m = \operatorname{lcm}(a, b)$, the **least common multiple** of $a$ and $b$ to be a number $m$ satisfying:
  - $m \geq 0$.
  - $a \mid m$ and $b \mid m$.
  - If $a \mid M$ and $b \mid M$ then $m \mid M$.

  Prove that
$$\operatorname{lcm}(a, b) = \frac{ab}{\gcd(a, b)}.$$