The Möbius Function and Möbius Inversion

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August Ferdinand Möbius (1790–1868) is perhaps most well known for the one-sided *Möbius strip* and, in geometry and complex analysis, for the *Möbius transformation*. In number theory, Möbius' name can be seen in the important technique of *Möbius inversion*, which utilizes the important *Möbius* function. In this PSP we'll study the problem that led Möbius to consider and analyze the Möbius function. Then, we'll see how other mathematicians, Dedekind, Laguerre, Mertens, and Bell, used the Möbius function to solve a *different* inversion problem.[1] Finally, we'll use Möbius inversion to solve a problem concerning Euler's totient function.

1 Möbius: the Möbius function

All excerpts of Möbius' work in this project are from *Uber eine besondere Art von Umkehrung der Reihen* (*On a special type of series inversion*). The following excerpt, from the beginning of Möbius' paper, sets up the basic form of Möbius' inversion problem:

> The famous problem of series inversion is that, when a function of a variable is given as a consecutive series of powers of the variable, one inversely requires the variable itself, or even any other function of it, expressed as an ongoing series of powers of the original function. One knows that it requires no small analytical ingenuity to discover the rule according to which the coefficients of the second series depend on the coefficients of the first. The following task is much easier to solve.
>
> Suppose a function $f(x)$ of a variable x is given as a series according to the powers of x:
>
> $$f(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots \quad (1)$$
>
> One should represent x as an ongoing series, not according to the powers of the function $f(x)$, but rather according to the function f of the powers of x:
>
> $$x = b_1 f(x) + b_2 f(x^2) + b_3 f(x^3) + b_4 f(x^4) + \dots \quad (2)$$

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[1] All translations of excerpts from the German works of Möbius, Dedekind, and Mertens in this project were done by David Pengelley, New Mexico State University (retired), 2021. The author of this PSP is responsible for the translation of the French excerpt from the work of Laguerre.

Task 1 Where have you seen a function written in the form in (1)?

Task 2 The expression in (2) is the inversion of the expression in (1). Why would this be called an *inversion*?

Möbius continued, and stated the goal of the problem:

The main demand of our problem is: Express the coefficients $b_1, b_2, b_3, \ldots$ of the series (2) as functions of the coefficients $a_1, a_2, a_3, \ldots$ of the series (1); and this occurs through the following very easy calculation.

Task 3 In your own words, what is the objective?

Ok, now it's time to get our hands dirty. Given that

$$f(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots,$$

we'll find expressions for $b_1, b_2, b_3, b_4, \ldots$ in terms of the given coefficients $a_1, a_2, a_3, a_4, \ldots$.

The symbolic equations have been removed from the following two excerpts. The tasks that follow ask you to fill in the missing sets of equations.

From (1) flows:

Equation Set A

If one substitutes these values of $f\left(x^2\right), f\left(x^3\right), \ldots$ and of fx itself from (1) into the equation (2), one gets:

Equation B

Task 4 Give expressions for $f(x^2)$, $f(x^3)$, $f(x^4)$, $f(x^5)$, and $f(x^6)$. These are Equation Set A.

Task 5 Do what Möbius instructed: “substitute these values of $f\left(x^2\right), f\left(x^3\right), \ldots$ and of $f(x)$ itself from (1) into the equation (2)” *and* then rearrange the expression you obtain on the right so that it's in the form

$$\_\_\_\_x + \_\_\_\_x^2 + \_\_\_\_x^3 + \_\_\_\_x^4 + \_\_\_\_x^5 + \_\_\_\_x^6 + \ldots. \tag{3}$$

This is Equation B.

Task 6 Give the coefficient of x^{23} in terms of a's and b's.

Task 7 Give the coefficient of x^{24} in terms of a's and b's.

Möbius continued:

The law of progression of the coefficients in this series is clear. Namely, to determine the coefficient of x^m, partition the number m in all possible ways into two positive whole factors. Each of these products then gives a term of the coefficient sought, in that one takes the two factors of the product as indices of an a and b to multiply together.

Because the equation above must hold for every value of x, we have:

Equation Set C

through which every b can be calculated with the aid of the previous b's.

In order therefrom to find the individual b's independently from one another, one sets $a_1 = 1$ for the sake of greater simplicity, and obtains:

Equation Set D

Task 8 Möbius explained how to obtain the coefficient of x^m in this expansion in the first paragraph of this excerpt. Compare his explanation to your answers *and* to your work for Tasks 6 and 7.

Remember, the expression you found, (3), is the right hand side of (2):

$$x = b_1 f(x) + b_2 f(x^2) + b_3 f(x^3) + b_4 f(x^4) + \ldots.$$

Task 9 Next, Möbius stated “Because the equation above must hold for every value of $x\ldots$” So, match (3) with the left hand side of (2) in order to obtain conditions on all the coefficients you found in (3). This will be a list of *equations* with a's and b's on the one side of the equality, and a number on the other. This is Equation Set C.

At this point, Möbius decided to let $a_1 = 1$ for convenience. We'll do the same. There's no harm done; if the function you are interested in doesn't have $a_1 = 1$ use the function $\frac{1}{a_1} f(x)$ instead and adjust accordingly in the end.

From Equation Set C, you can now find values for the b's in terms of the a's.

Task 10 What is b_1?

Task 11 Use the value of b_1 to find the value of b_2 in terms of a's. Continue: find b_3, b_4, b_5, b_6, b_7, and b_8 in terms of a's (no b's). These are Equation Set D.

Next, Möbius made an observation about how to form the b's without the need to extend the process above indefinitely:

> These few developments are sufficient to take away how also the values of the succeeding b's are put together from $a_2, a_3, \ldots$. Namely one decomposes the index m of b_m in all possible ways into factors, in which one takes m itself as the largest factor, but omitting 1, and also considers any two decompositions, that differ only in the order of their factors, as different; or as one can express briefly in the language of combinatorial theory: One builds all variations with repetition to the product m. Each of these variations then gives a term in the value of b_m, taking the elements of the variation as the indices of a's, and this term receives the positive or negative sign, according to whether the number of elements is even or odd.
>
> So for example all variations of the product 12 are:
>
> $$12,\ 2\cdot 6,\ 3\cdot 4,\ 4\cdot 3,\ 6\cdot 2,\ 2\cdot 2\cdot 3,\ 2\cdot 3\cdot 2,\ 3\cdot 2\cdot 2,$$
>
> and thus
>
> $$b_{12} = -a_{12} + 2a_2a_6 + 2a_3a_4 - 3a_2a_2a_3.$$
>
> The general correctness of this rule flows from the recurrence formula (Equation Set C) so easily that it would be superfluous for us to tarry for a proof.

Task 12 Use Möbius' observation to give an expression for b_8 in terms of a's. Compare both your answer and your process to those of Task 11.

Task 13 Use Möbius' observation to give an expression for b_{31} in terms of a's.

Task 14 Give an expression for b_{45} in terms of a's.