Week 8 Notes: 2019 October 14-16

MATH 465/565
Towson University

Monday, 2019 October 14

Theorem. *If $f(x)$ is multiplicative and $g(n) = \sum_{d|n} f(d)$, then $g(n)$ is multiplicative and vice-versa, so if $g(n)$ is multiplicative, then $f(n)$ is as well.*

Proof. Suppose $g(n)$ is multiplicative and assume $\gcd(m, n) = 1$. Then,

$$f(mn) = \sum_{d|mn} g(d).$$

Since m and n are relatively prime, we can separate each divisor d of mn into unique factors e and h such that e is a divisor of m and h is a divisor of n; notice that $\gcd(e, h) = 1$. So,

$$\begin{aligned} f(m,n) &= \sum_{e|m} \sum_{h|n} g(eh) \\ &= \sum_{e|m} g(e) \sum_{h|n} g(h) \\ &= f(m)f(n). \end{aligned}$$

So, f is also multiplicative.

Now, suppose $f(n)$ is multiplicative. Then, by Theorem 6-6 (p. 87)—this states that if two arithmetic functions $f(n)$ and $g(n)$ satisfy one of the two conditions: $f(n) = \sum_{d|n} g(d)$ and $g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$ for each n, then they satisfy both conditions—we have that

$$g(mn) = \sum_{d|mn} \mu(d) f\left(\frac{mn}{d}\right).$$

Hence, following an argument similar to that in the initial direction,

$$\begin{aligned} g(mn) &= \sum_{e|m} \sum_{h|n} \mu(eh) f\left(\frac{mn}{eh}\right) \\ &= \sum_{e|m} \sum_{h|n} \mu(e)\mu(h) f\left(\frac{m}{e}\right) f\left(\frac{n}{h}\right) \\ &= \sum_{e|m} \mu(e) f\left(\frac{m}{e}\right) \sum_{h|n} \mu(h) f\left(\frac{n}{h}\right) \\ &= g(m)g(n). \end{aligned}$$

So, g is also multiplicative. □

Example 1. $f(n) = n$ is multiplicative since $f(nm) = f(n)f(m)$ if $\gcd(m, n) = 1$.

Since $n = f(n) = \sum_{d|n} \varphi(n) \implies \varphi(n)$ is multiplicative.

$\sigma(n) = \sum_{d|n} d = \sum_{d|n} f(d) \implies \sigma$ is multiplicative.

The majority of class was spent examining one of Euler's first papers regarding Fermat's theorem (see worksheet).

Wednesday, 2019 October 16

Definition. Call a Gaussian integer $(a + bi) = z$ a **unit** if there exists another Gaussian integer w with $z \cdot w = 1$.

Ignore units when talking about unique factorization (same is true with any ring).

1 Eisenstein integers

This topic was covered in a talk given in class on this day. As a brief overview, we have that Eisenstein integers are of the form $\omega = e^{\frac{i\pi}{2}}$. The collection of Eisenstein integers can be described as $\{a + b\omega \mid a, b \in \mathbb{Z}\}$. Eisenstein integers have unique factorization.

Units in Gaussian integers are $\{1, -1, i, -i\}$; units in Eisenstein integers are $\{1, \omega, \omega^2 = \overline{-\omega}, -\omega, \overline{\omega}, -1\}$. The Eisenstein integers are defined over a triangular lattice.