Class Notes

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1.1 Analysis of $\pi(n)$

$\pi(x)$ is the count of the number of primes less than or equal to x.
Last time we talked about how $\lim_{x \to \inf} \pi(x) = \inf$

1.1.1 New Proof

Consider $\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \sum_{i=0}^{\inf} \frac{1}{2}^i$
$\frac{1}{1-\frac{1}{3}} = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ... = \sum_{i=0}^{\inf} \frac{1}{3}^i$
Multiply these together:
$(\frac{1}{1-\frac{1}{2}})(\frac{1}{1-\frac{1}{3}}) = 2(\frac{3}{2} = 3$
$(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...)(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...)$
$= (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = \sum_{\text{n is not divisible by any prime bigger than 3}} \frac{1}{n}$
$(\frac{1}{1-\frac{1}{2}})(\frac{1}{1-\frac{1}{3}})\frac{1}{1-\frac{1}{5}} = \sum_{\text{n is not divisible by any prime bigger than 5}} \frac{1}{n}$
$\Pi_{p \leq N}(\frac{1}{1-\frac{1}{p}}) = \sum_{\text{n is not divisible by any primes bigger than N}} \frac{1}{N} > \sum_{n \leq N} \frac{1}{n}$

Calculus 2 tells us the Harmonic series diverges so as $N \to \inf$, $\Pi_{p \leq N}(\frac{1}{1-\frac{1}{p}} \to \inf$
thus this product must have infinitely many primes to multiply

We want to use this idea to say more about $\pi(x)$.

1.2 Theorem

For any positive integer k, we have
$\frac{\pi(x)}{x} \leq \frac{\varphi(k)}{k} + \frac{2k}{k}$

1.2.1 Examples

Consider $k = 6$ $\varphi(6) = 2$ so all the primes besides 2,3 are either 1 or 5 (mod 6).
$\frac{\pi(x)}{x} \leq \frac{2}{6} + \frac{2(6)}{x}$
$\frac{\pi(x)}{x} \leq \frac{1}{3} + \frac{12}{x}$

1.2.2 Proof

Let $M = \lfloor \frac{x}{k} \rfloor$
Divide the interval $[1, x]$ into M intervals of length k and one final shorter interval.
We want to count the primes smaller than x.
Between 0 and k, there is at most k primes.
Between k and $2k$, there are k numbers forming a complete residue class (mod K).
Any prime p in this interval has $gcd(p, k) = 1$ so at most $\varphi(k)$ primes.
The last interval can't have more than k primes.
Total count $\pi(x) \leq k + \varphi(k) + \varphi(k) + ... + k \leq 2k + (M-1)\varphi(k)$
Note that $M = \lfloor \frac{x}{k} \rfloor$ so $M \leq \frac{x}{k}$ which implies $kM \leq x$, thus $k(M-1) \leq x$
So

$$\begin{aligned} \frac{\pi(x)}{x} &\leq \frac{2k + (M-1)\varphi(k)}{x} \\ &\leq \frac{2k}{x} + \frac{(M-1)\varphi(k)}{x} \\ &\leq \frac{2k}{x} + \frac{(M-1)\varphi(k)}{(M-1)k} \\ &\leq \frac{2k}{x} + \frac{\varphi(k)}{k} \end{aligned}$$

1.3 Theorem

$\lim_{x \to \inf} \frac{\pi(x)}{x} = 0$
The proportion of integers that are prime trends to zero as x approaches infinity.
Our goal is to show that for any $\epsilon > 0$, there exists and N so that if $x \geq N$, then $\frac{\pi(x)}{x} < \epsilon$

1.3.1 Proof

Fix $\epsilon > 0$
We want $\frac{\pi(x)}{x} < \epsilon$ for all $x \geq N$ for some N to be determined.
We know $\frac{\pi(x)}{x} \leq \frac{2k}{x} + \frac{\varphi(k)}{k}$
Goal: Show that $\frac{2k}{x} < \frac{\epsilon}{2}$ and $\frac{\varphi(k)}{k} < \frac{\epsilon}{2}$
Suppose $k = p_1 p_2 ... p_r$ is the product of the first r primes.

Then

$$\frac{\varphi(k)}{k} = \frac{k(1-\frac{1}{p_1})(1-\frac{1}{p_2})...(1-\frac{1}{p_r})}{k}$$
$$= (1-\frac{1}{p_1})(1-\frac{1}{p_2})...(1-\frac{1}{p_r})$$
$$= \Pi_{i=1}^{r}(1-\frac{1}{p^i})$$

Recall: $\Pi_{p<N}(\frac{1}{1-\frac{1}{p}} > \sum_{n<N}\frac{1}{n}$
$\Pi_{p<N}(1-\frac{1}{p})^{-1} > \sum_{n<N}\frac{1}{n}$ which looks like $\Pi_{i=1}^{r}(1-\frac{1}{p_i}$.
Take the reciprocal of both sides
$\Pi_{p<N}(1-\frac{1}{p} < (\sum_{n<N}\frac{1}{n})^{-1}$. This sum diverges.
Lets pick N_1 big enough so that $(\sum_{n<N}\frac{1}{n})^{-1} < \frac{\epsilon}{2}$
Then we pick p_r to be the smallest prime bigger than N_1
$\frac{\varphi(k)}{k} = \Pi_{p<p_r}(1-\frac{1}{p} < \Pi_{p<N}(1-\frac{1}{p}) < (\sum_{n<N_1}\frac{1}{n})^{-1} < \frac{\epsilon}{2}$
Pick $k = p_1 p_2 ... p_r$, then $\frac{\varphi(k)}{k} < \frac{\epsilon}{2}$. We are halfway there.

Now we want $\frac{2k}{x} < \frac{\epsilon}{2}$
As long as $\frac{2k}{\frac{\epsilon}{2}} < x$ and $N_2 = \frac{4k}{\epsilon} < x$ holds, then $\frac{2k}{x} < \frac{\epsilon}{2}$
Now we let $N = max(N_1, N_2)$
If $x > N$ then $\frac{\pi(x)}{x} \leq \frac{2k}{x} + \frac{\varphi(k)}{k}$
Using the k we chose,
$\frac{\pi(x)}{x} < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ so $\frac{\pi(x)}{x} < \epsilon$ which is what we wanted.

2 October 30

2.1 Analysis of $\pi(n)$

$\lim_{x\to\inf}\frac{\pi(x)}{x} = 0$
Most numbers are not prime

2.2 Prime Number Theorem

$\lim_{x\to\inf}\frac{\pi(x)}{\frac{x}{ln(x)}} = 1$

2.3 Chebyshev's Theorem

There exist constants c_1, c_2 such that
$c_1\frac{x}{ln(x)} < \pi(x) < c_2\frac{x}{ln(x)}$

2.3.1 Lemma

For any x, $0 \leq \lfloor 2x \rfloor - 2\lfloor x \rfloor \leq 1$

2.3.2 Proof

$x - 1 < \lfloor x \rfloor \leq x$
Multiply by 2:
$2x - 2 < 2\lfloor x \rfloor \leq 2x$
Replace x with $2x$:
$2x - 1 < \lfloor 2x \rfloor \leq 2x$
Bound $\lfloor 2x \rfloor - 2\lfloor x \rfloor$ below
$\lfloor 2x \rfloor - 2\lfloor x \rfloor > (2x - 1) - (2x)$
$\lfloor 2x \rfloor - 2\lfloor x \rfloor > -1$
Bound above: $\lfloor 2x \rfloor - 2\lfloor x \rfloor < 2x - (2x - 2)$
$\lfloor 2x \rfloor - 2\lfloor x \rfloor < 2$
Since $\lfloor 2x \rfloor - 2\lfloor x \rfloor$ is an integer strictly between -1 and 2, the only integer possibilitiies in this case are 0 and 1. Thus,
$0 \leq \lfloor 2x \rfloor - 2\lfloor x \rfloor \leq 1$

2.4 Theorem

The number of times that p divides $n!$ is $\sum_{i=1}^{\inf} \lfloor \frac{n}{p^i} \rfloor$

2.4.1 Example

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 = 2^3(3)(5)$
2 divides 120 3 times.

$$\sum_{i=1}^{\inf} \lfloor \frac{5}{2^i} \rfloor = \lfloor \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + ... \rfloor$$
$$= 2 + 1 + ... + 0 + 0 + ...$$
$$= 3$$

2.4.2 Proof

Note: If $p > n$, then $p \nmid n!$ and $\lfloor \frac{n}{p^i} \rfloor = 0$ for all i.
If $p \leq n$ then the number of integers up to n divisible by p is $\lfloor \frac{n}{p} \rfloor$
Each multiple of p contributes one factor.
Each multiple of p^2 contributes one more, thus there are $\lfloor \frac{n}{p^2} \rfloor$ such terms.
Each multiple of p^i contributes an additional factor, so there are $\lfloor \frac{n}{p^i} \rfloor$ such numbers.

2.5 Lemma

Let $f(x) = \frac{x}{ln(x)}$ then
$f'(x) > 0 for x > e$
$f(x - 2) > \frac{1}{2} f(x)$ when $x \geq 4$
$f(\frac{x+2}{2} < \frac{15}{16} f(x)$ when $x \geq 8$

2.5.1 Proof

$f'(x) = \frac{ln(x) - x\frac{1}{x}}{(ln(x))^2} = \frac{ln(x)-1}{(ln(x))^2} > 0$ if $x > e$
Note: if $x \geq 4$, $x - 2 \geq \frac{x}{2}$
$f(x-2) = \frac{x-2}{ln(x-2)} > \frac{x}{2ln(x-2)} > \frac{x}{2ln(x)}$
It can be seen that $\frac{x}{2ln(x)} = \frac{1}{2}f(x)$

2.6 Proof of Chebyshev's Theorem

Take $c_1 = \frac{ln(2)}{2}$ and $c_2 = 30ln(2)$
Goal: Prove that $(\frac{ln(2)}{2})(\frac{x}{ln(x)} \leq \pi(x) \leq 30ln(2)(\frac{x}{ln(x)}$
Trick: Consider middle binomial coefficient $\binom{2n}{n}$
$\binom{2n}{n} = \frac{2n!}{n!n!}$ which is an integer.
Every prime number bigger than n but less than $2n$, that is $n < p < 2n$ must divide $\binom{2n}{n}$
Let $P_n = \Pi_{n<p<2n} p$
Then $P_n | \binom{2n}{n}$
Thus $P_n \leq \binom{2n}{n}$
On the other hand, every prime $n < p < 2n$ is bigger than n so
$P_n = \Pi n < p < 2np > n^{\pi(2n)-\pi(n)}$ because $n < p < 2n = \pi(2n) - \pi(n)$
So $n^{\pi(2n)-\pi(n)} < P_n \leq \binom{2n}{n}$
Define r_p by $p^{r_p} \leq 2n < p^{r_p+1}$

Example: $n = 20, p = 5$
$5^{r_5} \leq 2(20) < 5^{r_5+1}$
$r_5 = 2$

The number of times that $p|(2n)!$ is $\sum_{i=1}^{r_p} \lfloor \frac{2n}{p^i} \rfloor$ where r_p captures the last item in this sum that is nonzero.
Same for $n!$
The number of times $p|n!$ is $\sum_{i=1}^{r_p} \lfloor \frac{n}{p^i} \rfloor$
The number of times $p|\binom{2n}{n} = \frac{2n!}{n!n!}$ is
$\sum_{i=1}^{r_p} \lfloor \frac{2n}{p^i} \rfloor - 2\sum_{i=1}^{r_p} \lfloor \frac{n}{p^i} \rfloor = \sum_{i=1}^{r_p} (\lfloor \frac{2n}{p^i} \rfloor - 2\lfloor \frac{n}{p^i} \rfloor$
By earlier lemma $0 \leq \lfloor 2x \rfloor - 2\lfloor x \rfloor \leq 1$
$\sum_{i=1}^{r_p} (\lfloor \frac{2n}{p^i} \rfloor - 2\lfloor \frac{n}{p^i} \rfloor) \leq r_p$
Define $Q_n = \Pi_{p<2n} p^{r_p}$
Then $\binom{2n}{n} | Q_n$ since every prime less than $2n$ appears more times in Q_n than it does in $\binom{2n}{n}$
Since $p^{r_p} < 2n$ then $Q_n = \Pi_{p<2n} p^{r_p} < \Pi_{p<2n}(2n)$
$\Pi_{p<2n}(2n) = (2n)^{\pi(2n)}$
$\binom{2n}{n} \leq Q_n < (2n)^{\pi(2n)}$