Math 565 - Fall 2019
Homework 7
Due November 6, 2019

There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, despite their simple definition and role as the building blocks of the natural numbers, the prime numbers grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision.

— Don Zagier

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(1) Modify Euclid's proof that there are infinitely many primes to show that there are infinitely many primes congruent to 3 (mod 4).
(2) Problem 8-1.9 (Hint: Your proof should show that $p_n < 2^{2^n} + 1$. In the induction step, you can use that if $p_n < 2^{2^n} + 1$, then $p_n \leq 2^{2^n}$.)
(3) Problem 8-2.1
(4) Prove that $n! + k$ is divisible by a prime greater than n for every k where $1 \leq k < n$.