Graph Theory

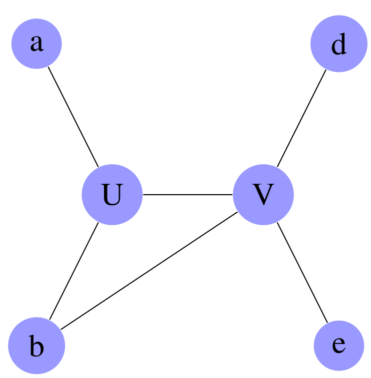
Christina Donall

May 8th, 2018

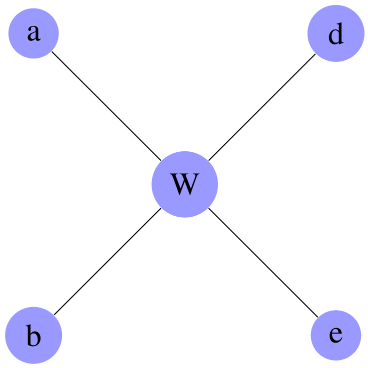
From last class: It was possible to draw k_5 on the torus. $\gamma(k_5) = 1$. (γ - number of holes required to draw a graph on a surface.) $\gamma(k_6) = \gamma(k_7) = 1$. $\gamma(k_8) = 2$.
Def: A graph minor of a graph G is a graph obtained from G by
1) Deleting edges
2) Deleting vertices
3) Contracting an edge

To contract an edge in a graph G (Let's call the edge UV), we replace both vertice U and V with a new vertex that is connected to every vertex that either U or V was previously connected to.

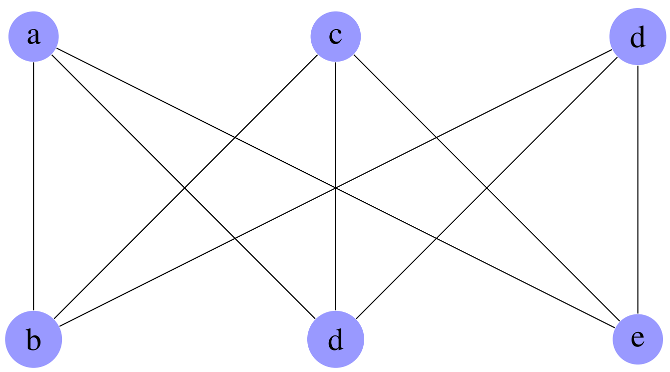
Example: Contracting an edge



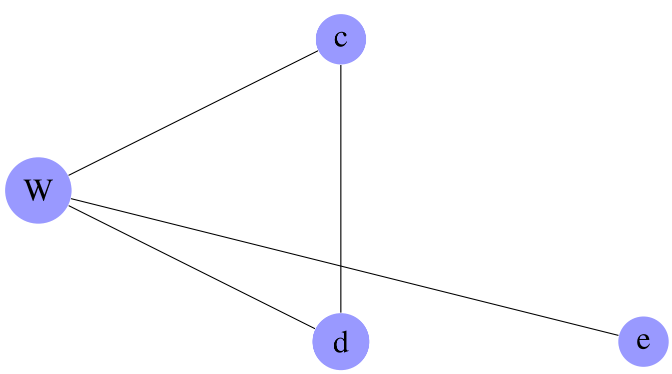
Contract edge UV



Example: Graph Minor. Below is a bipartite graph.



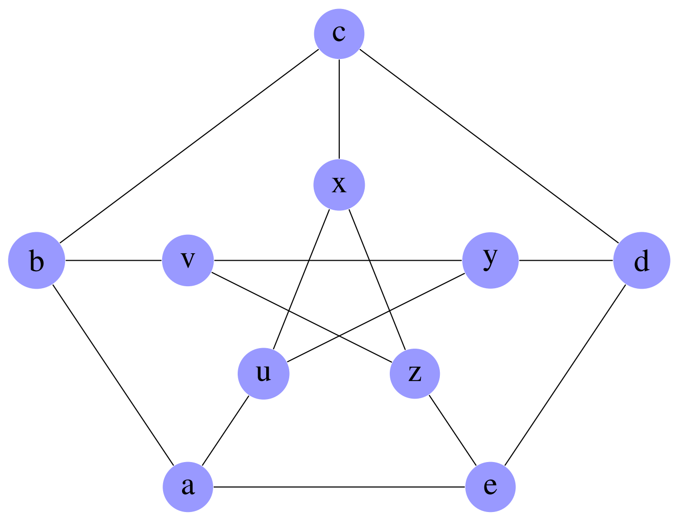
Contact edge ab, Delete edge ce, Delete vertex d. Which changes the bipartite graph to a not bipartite graph.



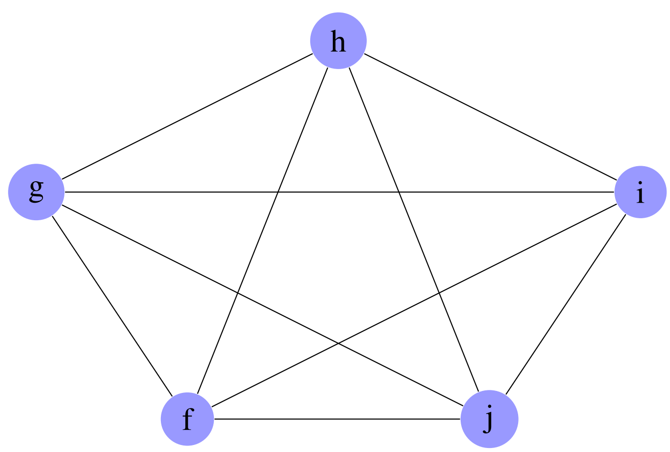
Graph Minor Theorem: In any infinite list of graphs, there will always be one graph that is a minor of another graph in the list.

Wagner's Theorem: A graph G is planar if and only if it does not have k_5 or $k_3, 3$ as a graph minor.

Example: Peterson Graph is not planar.



Contract edges au, bv, cx, dy, ez to create k_5.



Theorem: For any surface, there exists a finite list of graphs such that a graph G can be drawn on the surface if and only if it does not contain one of the graphs from this list as a graph minor.

We still donâĂŹt know a complete list for the torus. We know that it contains at least 800 graphs. ■

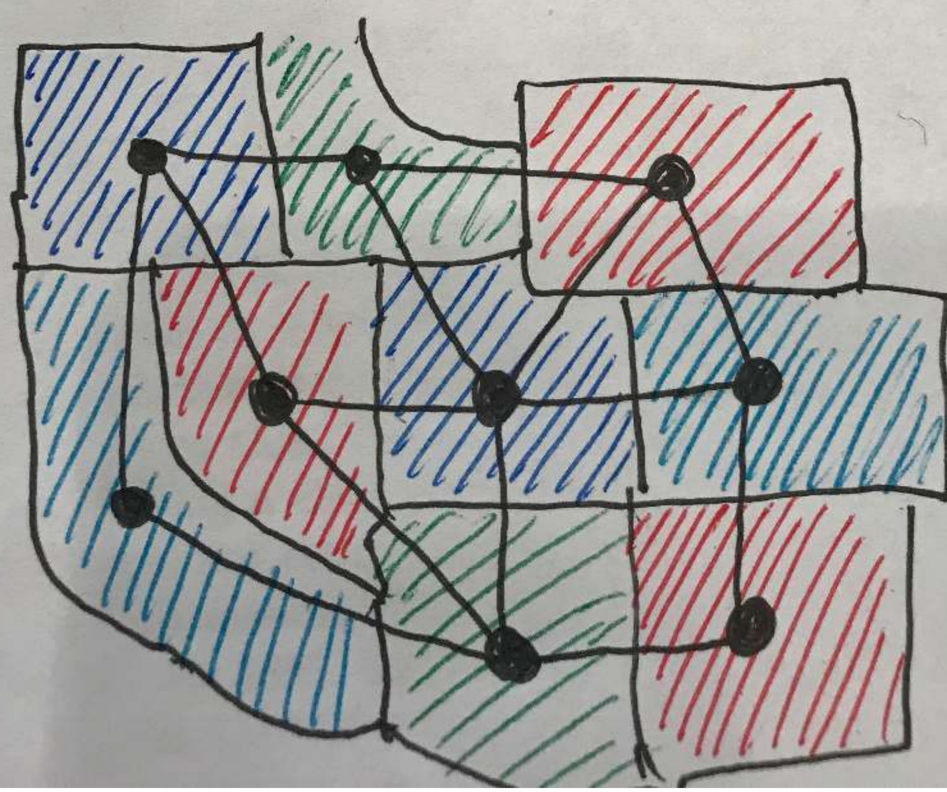
Coloring!! 10.1

Four Color Problem
In 1852, Francis Guthrie noticed that every map of the counties of England/Scotland/Whales could all be colored with 4 colors so that no two bordering countries had the same color.

Question of whether any map can be colored using 4 colors was not solved until 120 years later and only proved then relying on a super computer.

What does this have to do with graph theory?
Draw a map in the plane and put one vertex in each country. Connect vertices if the countries share a border (more than just a single point).

Western United States



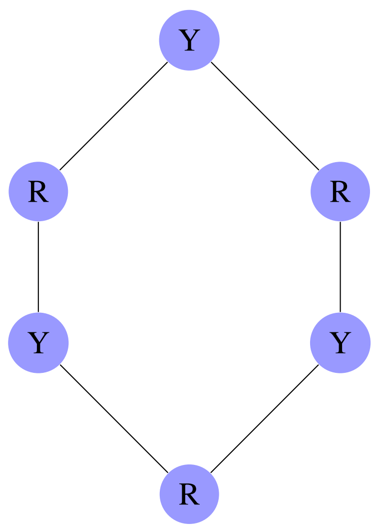
Def: For any graph G, we can define the chromatic number $\chi(G)$ to be the minimum number of colors required to color the graph G. (without adjacent vertices having the same color).

Trivial Graph: $\chi(\cdot) = 1$

If G has at least one edge, then $\chi(G) \geq 2$

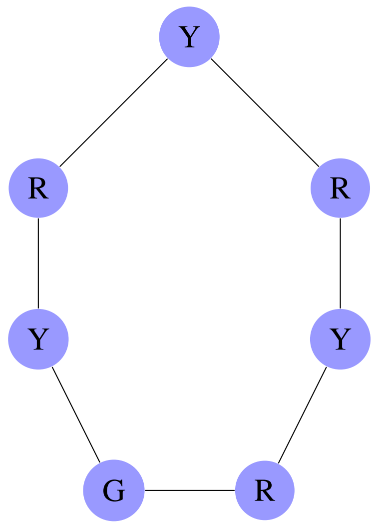
Note that for even n: $\chi(C_n) = 2$

Example: C_6 (R = red; Y = yellow)



Note that if n is odd, this does not work. If n is odd: $\chi(C_n) = 3$

Example: C_6 (R = red; Y = yellow; G = green)



A graph has chromatic number ≥ 2 if and only if it is bipartite.

$\chi(k_n)$ = n
$\chi(k_5) = 5$
$\chi(k_3, 3) = 2$ (bipartite)

1 ≥ χ(G) ≥ n

Proof: If it is bipartite, color is half a different color. If it is not bipartite, it contains an odd cycle and so requires at least 3.

Theorem: χ(G) ≥ 1 + Δ(G)

Proof: Number the possible colors 1,2,...k and then list the vertices of G as v_1, v_2, ... v_n. Now we take each vertex one at a time assign it the smallest color not already assigned to one of its neighbors. Since no vertex has more than Δ(G) neighbors, it is not possible for any vertex to be adjacent to vertices assigned all of the colors 1,2,...k (k = 1 + Δ(G)) Worst case all of a vertices neighbors has different colors. In this case, give the vertex the last color Δ(G) + 1 ■