Graph Theory Class Notes

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1 When can a graph be drawn on a 2d-plane so that no two edges cross?

Euler's Identity: IF a graph is drawn in the plane with n vertices and m edges and it divides the plane into r regions, then $n - m + r = 2$

Any time we can draw a graph in the plane we can also draw it on a sphere (and vice-versa)

Idea: Draw the graph on the plane

One to one map of points onto the sphere (except the north pole) to the plane

Using Euler's Identity we prove that if G is a planar graph with n vertices, and m edges then $M <= 3n - 6$

Corollary: Any planar graph mnust have at least wone vertex of degree less than 6.
Proof: Suppose for contradiction that such a graph exists (planar, every vertex has degree of at least 5). First theorem of graph theory tells us that $2m = \sum_{v \epsilon G} deg(v) >= \sum_{v \epsilon G} 6$
So $2m >= 6n$
$m >= 3n$
So m¿= 3n, but for a planar graph we must have m¡= 3n-6, which is a contradiction. This means k_7 is not planar (Every vertx has degree 6)

Theorem: k_5 is not planar
Proof: k_5 has n= 5, m = 10
check: 10 > 9 = (3 * 5 - 6)
So inequality m ¡= 3n-6 does not hold and k_5 cannot be drawn in teh plane
Note: k_6 is also not planar because it has k_5 as a subgraph

Preposition: If G has a subgraph H that is not planar then G is also not planar

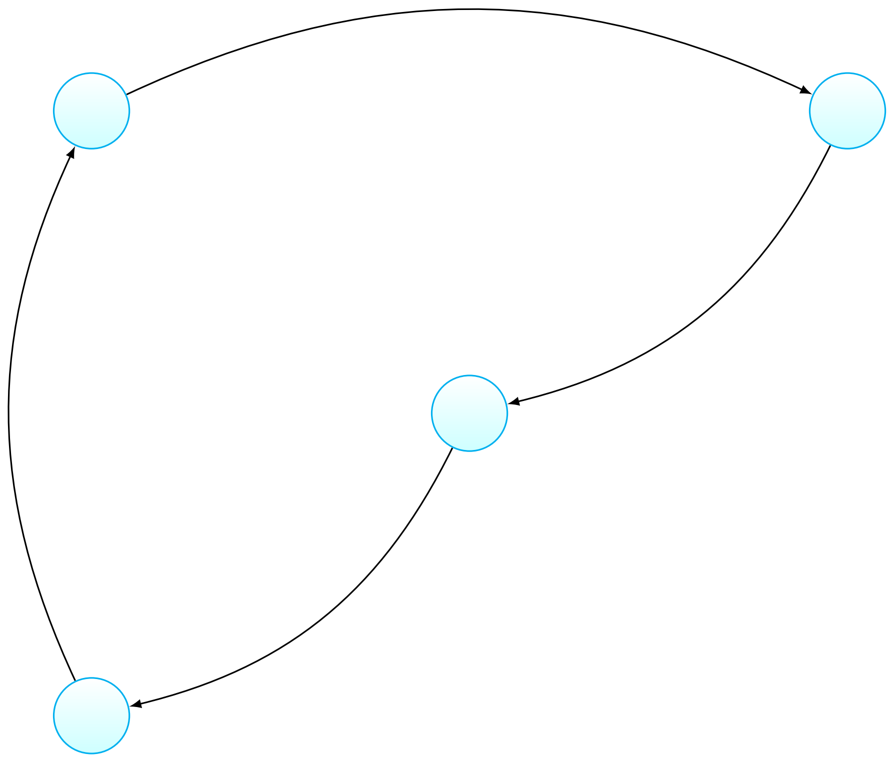
Can we use this to prove $K_{3,3}$ is not planar? (The 3 utilities problem)
n = 6, m = 9

Take $3(n) - 6 = 18 - 6 = 12 \neq 9$
This graph doesn't have too many edges, so to prove that $K_{3,3}$ is not planar we have to work a little harder
What if $K_{3,3}$ was planar? How many regions would it have?
Euler's identitiy sayas n - m + r = 2
6-9+r = 2
r=5
If $K_{3,3}$ could be drawn in the plane it must have 5 regions
$K_{3,3}$ does not have any 3 cycles in it, so none of the regions can be triangles
say these 5 regions are surrounded by m_1, m_2, m_3, m_4, m_5 edges each
$K_{3,3}$ doesn't have any bridges so
$2m = \sum_{i=1}^{5} m_i >= \sum_{i=1}^{5} 4 = 5(4) = 20$
since 2m = 18 is not greater than 20, $K_{3,3}$ cannot be planar

Summarize: Both k_5 and $K_{3,3}$ cannot be planar
Any graph with one of these as a subgraph also cannot be planar

Say H is a subdivision of G if H can be obtained by adding extra vertixes of degree 2 along the edges of G

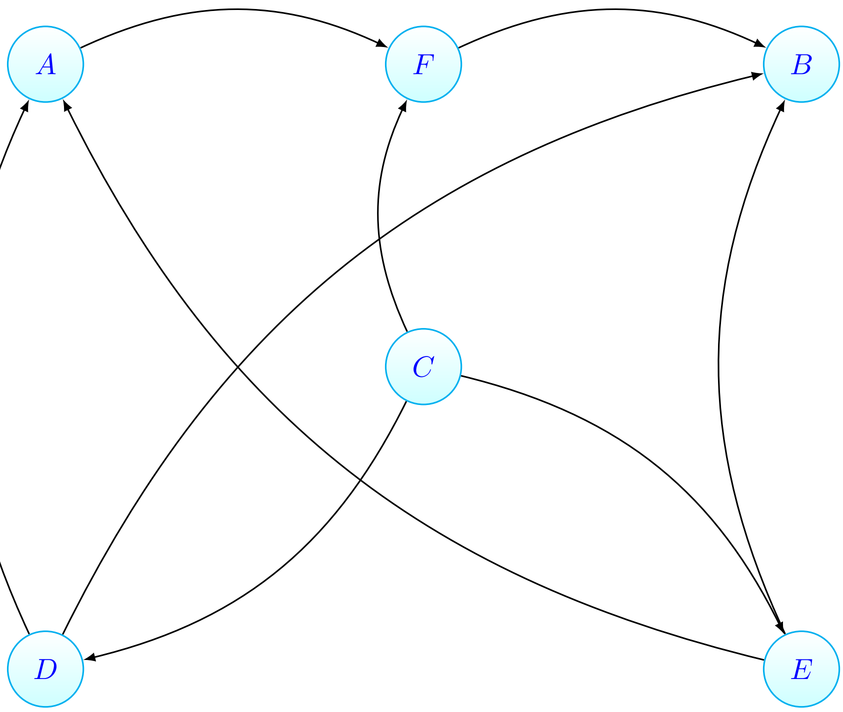
Ex: let G be K_3
then H =



is a subdivision of G

Kuratowski's Theorem: A graph G can be drawn in the plane without crossing edges if and only if it does not have a subgraph which is a subdivision of either K_5 or $K_{3,3}$

Ex: Prove that the peterson graph is non planar
There is no way that a subgraph of the peterson graph could be a subdivision of K_5 because the peterson graph is 3 regular and K_5 is 4 regular



$K_{3,3}$ is a subdivision of this subgraph of the peterson graph, so the peterson graph is not planar

The 5 sons problem could be solved with a single bridge (not a graph theoretic bridge) Adding a bridge is the same thing topologically as drawing the graph on a torus (donut) instead.

For any graph yuou can draw it on some surface as long as you have enough holes.

Call the number of holes in a surface the genus of the surface

For any graph G we can ask what is the minimum genus required to draw the graph without crossings?
Denote this by $\gamma(G)$
$\gamma(k_4) = 0$
$\gamma(k_5) = 1$
$\gamma(k_{3,3}) = 1$