Notes 5-10-2018

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Theorem: $\chi(G) <= 1 + \Delta(G)$
If $G = k_n$
$\Delta(kn) = n - 1$
$X(kn) = n - 1$
This theorem is sharp.
We can also consider C_n (when n is odd)
$\Delta(Cn) = 2$ $\chi(Cn) = 3$ (n is odd) Brook's Theorem: If G is a connected graph that is not k_n or C_n then $\chi(G) <= \Delta(G)$

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Lower Bounds: Def: $\omega(G)$ = Order of the largest Clique (every two distinct vertices in a clique are adjacent)

Def: $\chi(G)$ = Chromatic Number (The minimum number of colors required so that no two adjacent vertices share the same color)
$\chi(G) >= \omega(G)$
Note that $\chi(G)$ is often bigger than $\omega(G)$
Ex: C_n with odd n and $n >= 5$
$\omega(C_n) = 2$
$\chi(C_n) = 3$

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Grotszch Graph (See Figure 1.1) W(Gr) = 2 X(Gr) = 4

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Note that 3 colors are required to color the outside 5 vertices.

If there were a way to color the shadow vertices with 2 colors then we could give the outside vertices the same colors as their shadow.

So, the shadow vertices require 3 colors too.

Middle vertex has to use a new 4th color to be different from all of the shadows.

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//TODO: Get Table From Book $\alpha(G)$ = the size of a maximum independent set of vertices (No 2 vertices are adjacent to eachother)

$\beta(G)$ = minimum size of a vertex cover.

Vertex cover is a collection of vertices where every edge is adjacent to at least one vertex in the cover.

Theorem (HW) $\alpha(G) + \beta(G) = n$ for any graph G without isolated vertices.

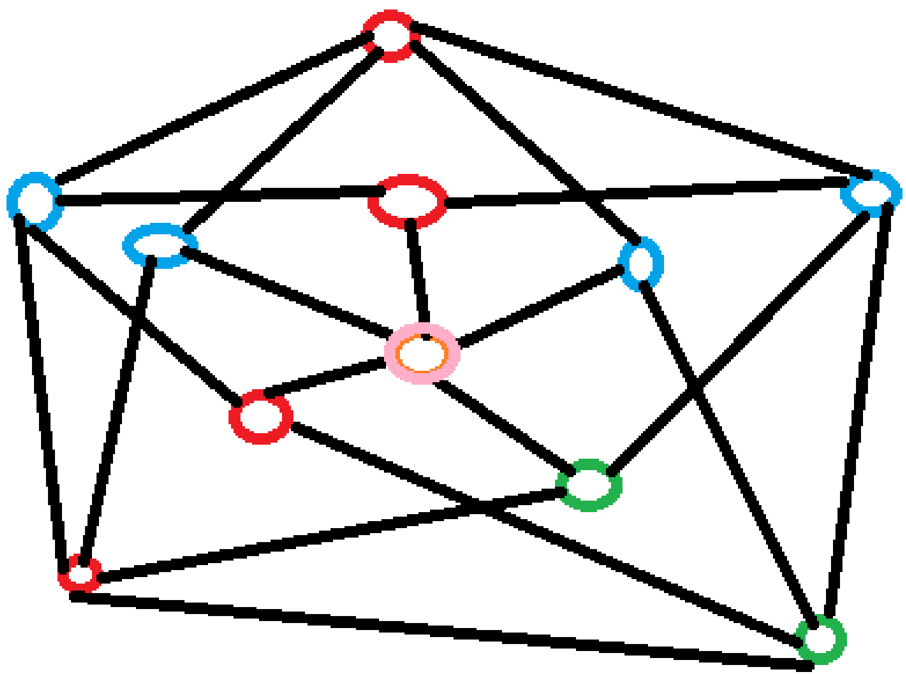

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Figure 1: Fig. 1.1: Grotszch Graph

Every set of vertices with the same color in a graph coloring form an independent set.

In a graph G of order n: $\chi(G) >= n/\alpha(G)$

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The 5-color map theorem is similar to the 4-color problem which was insufficiently proven by Alfred Kempe. Percy John Heawood found a counterexample, and subsequently expanded on Kempe's ideas. He proved it for 5 colors instead of 4.)

Theorem: If G is a planar graph then $\chi(G) <= 5$

Proof: Suppose this is false. Then there exists graphs drawn in the plane requiring more than 5 colors.

Let G be a graph requiring 6 colors of minimal order So any planar graph with fewer vertices can be colored with 5 colors.

Recall that any planar graph has a vertex of degree at most 5.

Let vertex v be the vertex in G that has the highest degree.

G-v is still planar and smaller than G, so G-v can be 5-colored by our assumption.

Case 1: If v has degree less than 5, then the neighbors of v only use at most 4 colors so we can add v back in and give it the remaining color.

So, we only need to worry about the case when deg(v) = 5 and all the neighbors of v have different colors in G-v.

Take 2 colors not adjacent to eachother, say green and blue, consider all of blue's green neighbors, and blue's green neighbor's blue neighbor's, and so on until there's a sequence of alternative green-blue all the way to the green connected to v.

Consider all of the vertices colored green or blue connected to the initial green and blue vertices.

If there exists a walk between the green and blue vertices, consisting only green/blue vertices, call this a Kempe chain. See Figure 1.2 for an example.

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Case 1: ((Suppose there is no Kempe chain; Then you can take a similarly generated path starting with one of those nodes and flip all greens to blues and blues to greens.))

If a Kempe chain did not exist, then we could change the color of the blue vertex to be green change its green neighbors to be blue, change their blue neighbors, etc, etc.

This still results in a valid coloring of the graph because this operation gives v two green neighbors, but no blue neighbors, so v can be blue.

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Case 2: Suppose the Kempe Chain does exist;

We'll prove it next time, but basically we can't get my from orange/pink color to the red or yellow colors, so there's no possible Kempe chain connected to that vertex, and see Case 1.

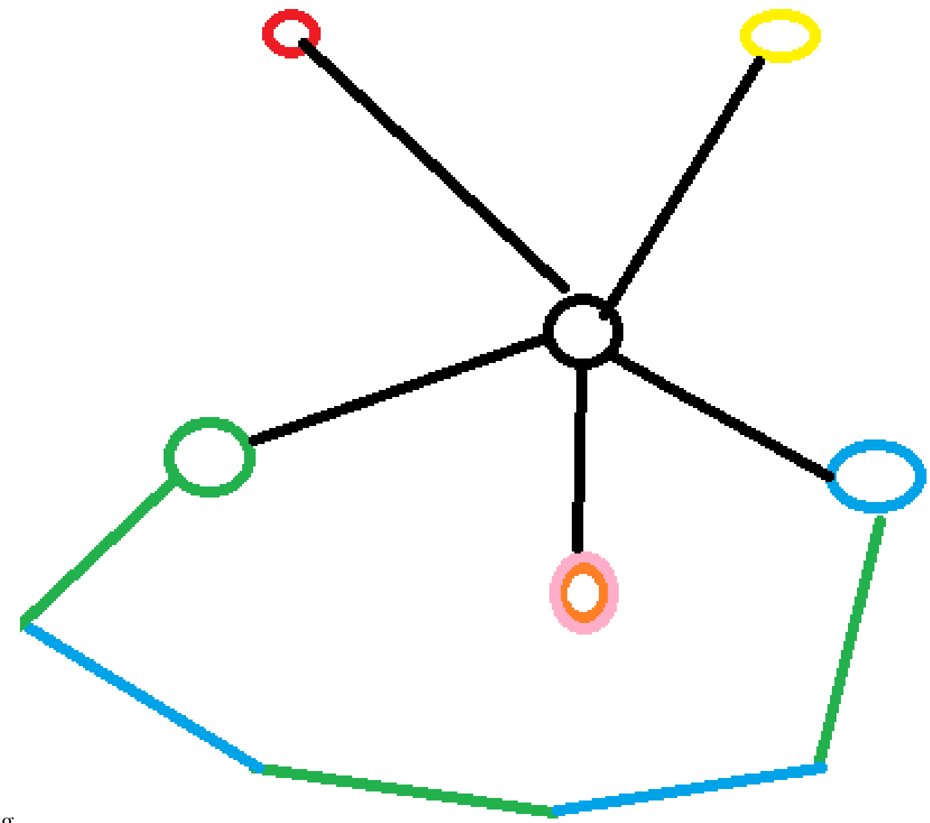

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Figure 2: Fig. 1.2: Kempe Chain