Class Notes 4/5

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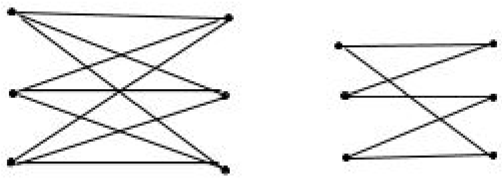
Last Time: Saw that the dodecahedron graph is Hamiltonian

1 Todays Notes

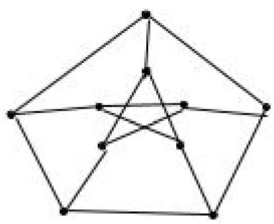
Observation: K_n is Hamiltonian for all n ≥ 3 , A complete bipartite graph of the same size $K_{n,n}$ is Hamiltonian.

Example: $K_{3,3}$



A Hamiltonian graph of $K_{3,3}$.

1.1 The Peterson graph is not Hamiltonian



Peterson Graph

1.2 Proof that Peterson graph is not Hamiltonian

Pf// The Peterson graph has

15 edges

5 outside

5 pentagram

5 connection edges

If there was a Hamiltonian cycle it would need to get from the outside to the inside and back again. Thus, the cycle uses either 2 or 4 connection edges.

Case 1: There are 2 of the connection edges used.

In order to connect all the vertices on the outside, these two edges must connect to adjacent vertices on the outside. Any two edges connected to adjacent vertices on the outside are not connected to adjacent vertices on the inside. So, there is not way to go through all the vertices on the inside.

Case 2: The 4 connection edges used

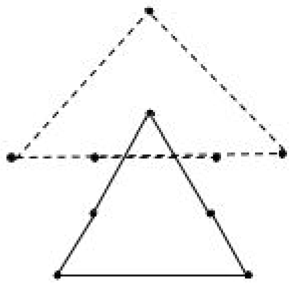


Figure 1.2.1

WLOG: Assume the top connecting edge is the one that is not used. Drawing the picture (Figure 1.2.1) and connecting all the edges that must be used in order to go through all the vertices, we are forced to have two 5 cycles, not on 10 cycle. //

1.3 Observation

If G has a cut-vertex then G is not Hamiltonian

1.4 Theorem

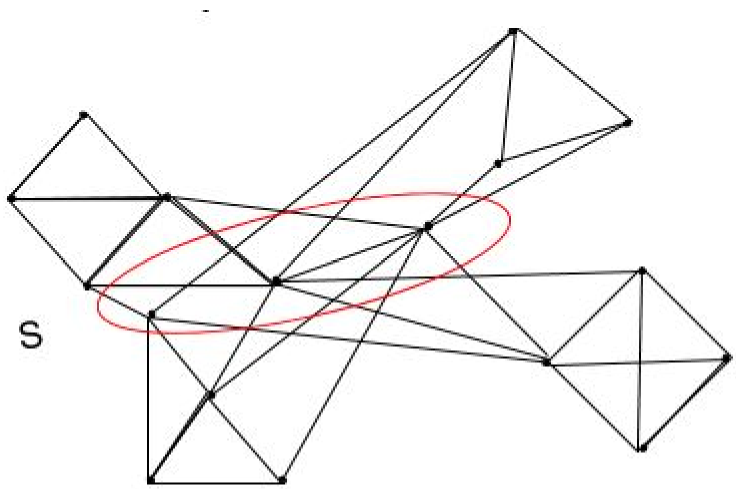
If G is Hamiltonian and S is any subset of the vertices of G.
Then K(G-S) $\leq$ |S|

Pf// Let G be a Hamiltonian Graph and S any subset of vertices.

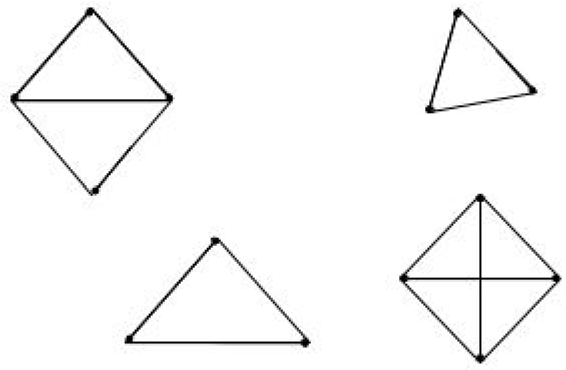
Let K= K(G-S).

This means that G-s has connected components G_1,G_2,....,G_k. Because G is Hamiltonian there exist a Hamiltonian cycle C. Now, for each component G_i there is some vertex that C goes through last before leaving. Then the next vertex of C after this one must be an element of S. Thus there exist a unique element of S that comes after each component G_i of G-S.
Then, |S| $\geq$ K(G-S). //

1.5 Example



Now we see G-S



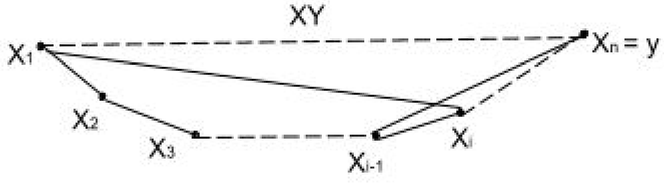
We can see |S| = 3 and K(G-S) = 4. Therefore we can say G is not Hamiltonian.

2 Ore's Criterian

2.1 Ore's Criterian

If G is a graph such that any two non-adjacent vertices u and v satisfy deg(u)+deg(v) $\geq$ n, then G is Hamiltonian.
Note: This is not an if and only if statement

Pf// Suppose this is false for contradiction.
Then there exist a graph G that is not Hamiltonian but any two non-adjacent vertices u and v of G satisfy deg(u)+deg(v) $\geq$ n.
Add edges to G so long as adding them does not make the graph Hamiltonian. At some point we get to a graph H, which has G as a subgraph and that is still not Hamiltonian but adding anymore edges to H does make it Hamiltonian.
There still exist two vertices, x and y, that are not connected in H but adding the edge xy would make H+xy Hamiltonian.
Note: That deg_H(x)+deg_H(y) $\geq$ n
Since H+xy is Hamiltonian it has a Hamiltonian cycle C.
C = { x = x_1,x_2,x_3, ... ,x_n = y}
Now Suppose that x_1 = x is connected by an edge to some other vertex x_i where 2 $\leq$ i $\leq$ n. Then claim that x_{i-1} is not connected to x in H.
Suppose it were:



Then (x_1,x_2, ... ,x_{i-1},x_n,x_{n-1},x_{n-2}, ... , $x_i x_1$) forms a Hamiltonian cycle in H. There are no Hamiltonian cycles in H.
Thus, if x is connected to x_i then y is not connected to x_{i-1}. This means that for every vertex x is connected to (x_i) there is a corresponding vertex (x_{i-1}) that y is not connected to.
The deg_H $\leq$ n-1-deg_H(y)
So, deg_H(x)+deg_H(y) $\leq$ n-1
This is our contradiction. //