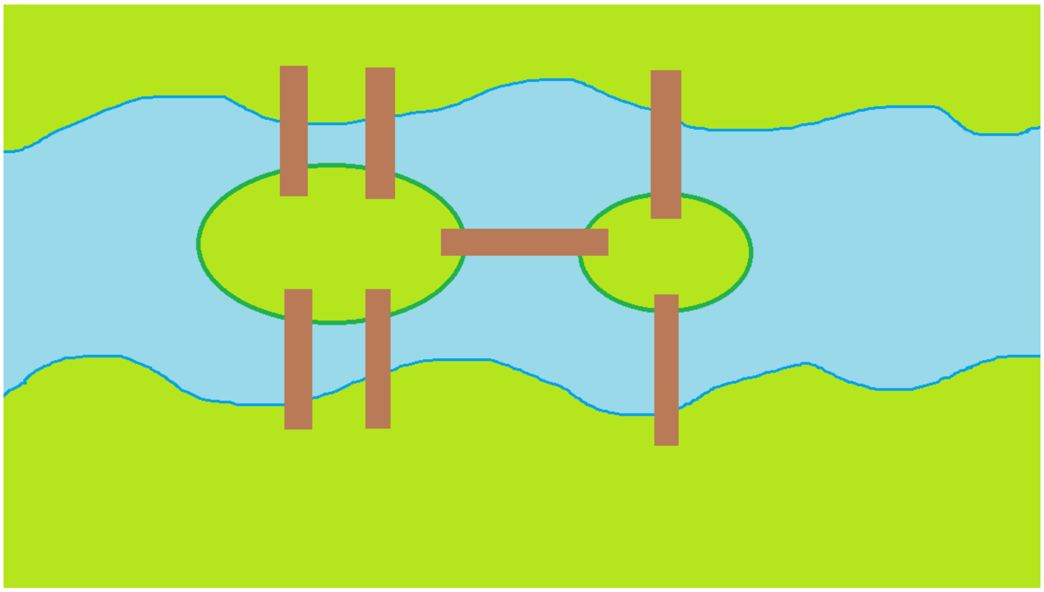
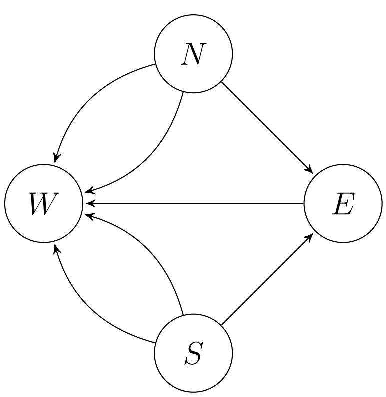


Julianne Jaeger
MATH 451
4/3/2018

Notes from 4/3

Map of Königsberg:

Multigraph of Königsberg:



Euler discovered that the reason why it was not possible to cross all the bridges of Königsberg without repeating one was that each land mass has an odd number of bridges (edges) connected to it.

Every place in between the starting point and ending point would need to have even degree for this to be possible.

An **eulerian trail** or **euler trail** in a graph is a trail where every edge in the graph is used once.

Recall that a trail never repeats edges.

An **eulerian circuit** or **euler circuit** is a circuit in a graph that uses every edge once.

Recall that a circuit is a trail where the end point is the same as the start point.

Euler's observation: To have an eulerian circuit, a graph cannot have any vertices of odd degree, and to have an eulerian trail, a graph must either have two or no vertices of odd degree.

A graph G is **eulerian** if it has an eulerian circuit.

Ex.

Theorem: A graph G is eulerian if and only if it is connected and every vertex has even degree.

Proof:

Suppose a graph G has an eulerian circuit. Then there exists a circuit passing through each vertex that uses all the edges in the graph. Since this circuit must both enter and leave each vertex once every time a vertex is visited, each vertex has even degree.

To show the other direction, suppose that G is connected and every vertex has even degree. We want to show that G has an eulerian circuit.

Pick a maximal trail in G by startng at any vertex u and continuing along unused edges for as long as possible until you reach a point where every edge has been used.

Claim: Once we get stuck, we must have ended back at u again.

To see this, note that if we get stuck at some other vertex $v \neq u$, then we entered this vertex but never left, so we used an odd number of edges leaving v. Since v has even degree, there must be another edge we haven't used yet.

So this longest possible trail is a circuit. Call it C.

It now remains to show that C contains every edge in the graph.

Consider the graph G-C, the graph with all the edges in C removed.

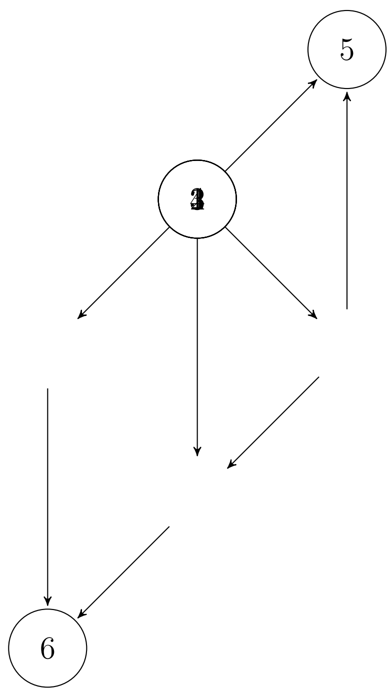


Figure 1: Has an eulerian circuit, is eulerian

Suppose for contradiction that there exists an edge e in G that is not in C.

Consider the connected component of the graph G-C containing the edge e.

Every vertex in this connected component had even degree in G, and an even number of edges connected to each vertex were part of C, so every vertex still has even degree in G-C.

Pick a maximal trail in this connected component, starting with e.

By our claim, this maximal trail must also be a circuit consisting entirely of edges not in C.

Now we can make our original circuit bigger by walking until we reach a vertex in this new circuit, following all the edges of this circuit, returning to this same vertex, and finishing the original circuit.

Thus, a longest possible circuit in the graph uses all of the edges in the graph, and hence it is eulerian.

Theorem: A graph G has an eulerian trail that is not a circuit if and only

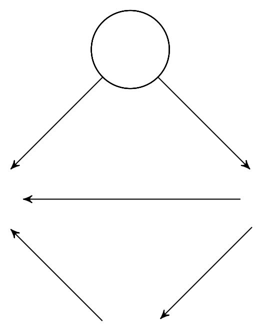
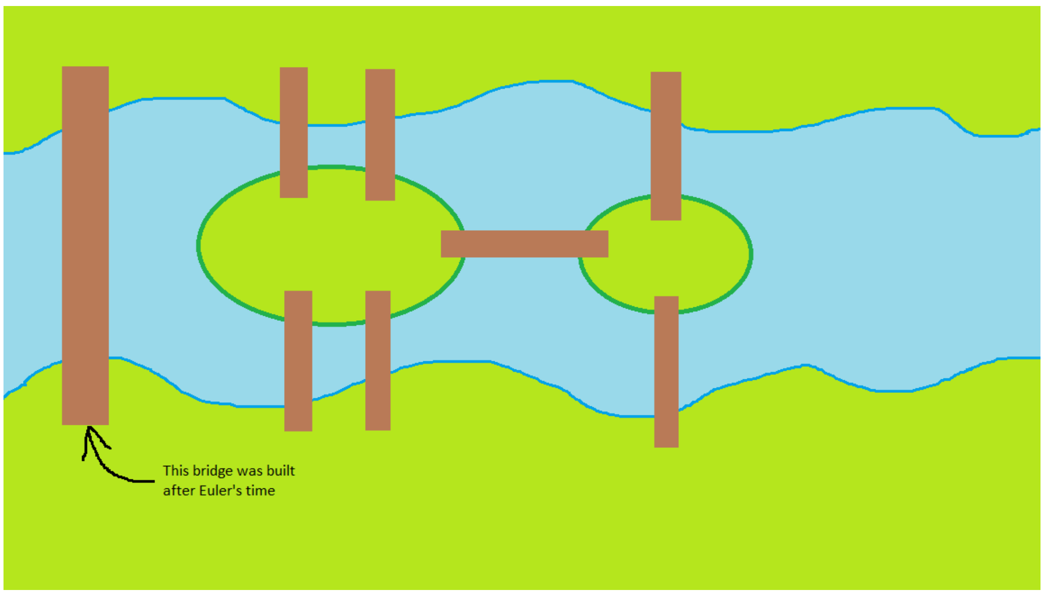


Figure 2: Has no eulerian circuit, is not eulerian



if the graph is connected and has exactly 2 vertices with odd degree.

Note: The trail must start and end with the vertices with odd degree.

Proof: In the book, similar to proof above

New map of Königsberg:

Multigraph of this new map:

We can consider the same problem for vertices instead of edges.

A path in a graph G that visits every vertex exactly once is a **Hamiltonian path**.

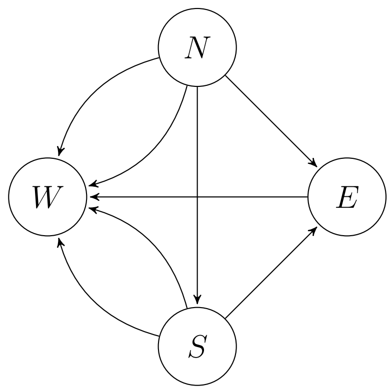


Figure 3: There are now exactly two vertices with odd degree, thus the graph contains an eulerian trail.

A **Hamiltonian cycle** is a cycle that uses every vertex exactly once.

A graph is **Hamiltonian** if it contains a Hamiltonian cycle.

Hamilton's Icosial Game: Find a way to walk along the edges of a dodecahedron visiting each of the vertices exactly once.