Graph Theory

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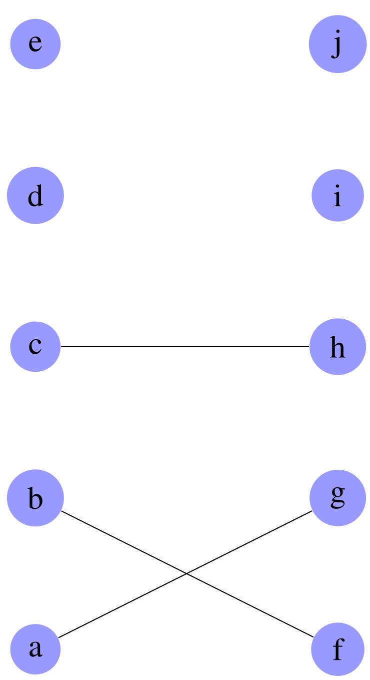
Recall from last class, we ended with the first half of the proof for Hall's Matching Theorem. We will start by continuing that proof.

Proof: Case 2: Otherwise, there exists some subset $x \in U$ with $|X| = |N(X)|$. Note that $|X| > |U| = k + 1$ and every subset of X satisfies the conditions of the theorem so by the induction hypothesis, we can find a matching from X to N(X)



This picture is true for any subset of our X(right) and out N(X)(left) of vertices.

We still need to show a matching on the vertices not in X. From U - X to V - N(X).

Since $|X| \geq 1$, this means that $|U - X| \leq k$. We need to show that U - X still satisfies the hypothesis of the theorem. We need to show that for every $S \subset U - X$, that $|S| \leq |N(S) - N(X)|$. We will call S' = N(S) - N(X). By the original assumption of the theorem, $|N \cup X| \leq |N(S \cup X)|$.

S and X are disjoint so,
$|S| + |X| = |S \cup X| \leq |N(S \cup X)| = |N(S) \cup N(X)|$
$= |S' \cup N(X)|$
$= |S'| + |N(X)|$

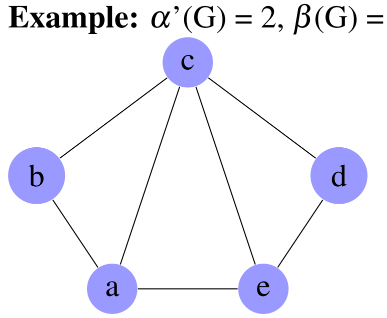
Since $|X| = |N(X)|$, we have that $|S| \leq |S'|$. So $G - (X \cup N(X))$ is a bipartite graph that still satisfies the conditions of the theorem. By the induction hypothesis, there exists a matching.

So by combining this matching with the one from X to N(X), we get a matching from U to V. ■

We can still talk about matchings for graphs that are not bipartite. In this case, a matching is often called an independent edge set, meaning a subset of the edges where no two edges have a vertex in common.

Def: A maximal independent edge set is an independent edge set of maximal size. We denote this by α'(G).

Def: An edge cover of G is a subset of edges where every vertex is part of at least one edge. β'(G) is the size of the minimal edge cover of G.
Example: α'(G) = 2, β(G) = 3



Test these two definitions on K_n, C_n, S_n

$\alpha'(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$,$\beta(K_n) = \left\lceil \frac{n}{2} \right\rceil$,$\alpha'(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$,$\beta(C_n) = \left\lceil \frac{n}{2} \right\rceil$,$\alpha'(S_n) = 1$,
$\beta(S_n)$ = n - 1.

Thm: If G is a graph of order n with no isolated vertices, then α'(G) + β'(G) = n.

Proof: Suppose G is a graph with no isolated vertices. First, supposeα'(G) = k. So there exist k independent edges in the graph. 2k of the vertices are covered by these edges. The remaining n-2k vertices can be covered by picking one edge for each vertex
so β'(G) $\leq$ k + (n - 2k)
so α'(G) + β'(G) $\leq$ k + k + (n - 2k) = 2k + (n - 2k) = n
so α'(G) + β'(G) $\leq$ n.

It remains to show that α'(G) + β'(G) $\geq$ n.

Suppose that β'(G) = ℓ and X is an edge cover of G of size ℓ. Take the subgraph of G induced by X. F = G[X], so F has order n.

Observation: F can not have any trail of length 3 or more.

Suppose it did, then the middle edge is redundant. We can remove it and still have an edge cover. So F has no cycles, so F is a forest. Any forest with n - k edges has k components, so ℓ = n - (n - ℓ) (size of F). So F must have n-ℓ components.

Pick one edge from each component of F. These edges are all independent.
α'(G) $\geq$ n - ℓ
so α'(G) + β'(G) $\geq$ n - ℓ + ℓ = n. ■