Notes from 4/19

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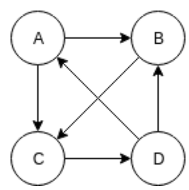
1 Chapter 7

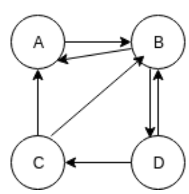
The transition matrix of a digraph is the matrix formed by creating a n x n matrix in which position ij contains the entry od(i) if there is an edge from i to j and 0 otherwise.
Example 1:



has the following transition matrix:

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

If we square T, we get a matrix of the probabilities of stepping from vertex i and being at vertex j 2 steps later.
If we raise this matrix to larger numbers we see that the columns trend toward the same values. That tells us that for many steps, where you start does not matter for this graph.
Example 2:



has the following transition matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

If we raise this transition matrix to higher powers, the columns do not trend toward certain values. Instead, the columns alternate between sets of values on successive exponents.
This is because if we start in $U = A, D$ then after an even number of steps, we are in U and after an odd number of steps, we are in $V = B, C$.

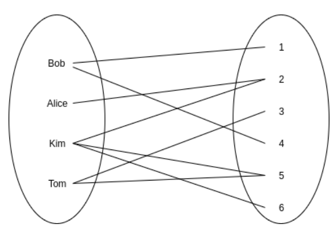
Definition: A digraph is called ergodic is there exists an integer N such that for every $n > N$, there is a walk of length n between any 2 vertices of the digraph.
(N is the minimum number of steps before it is possible to get between any 2 vertices in exactly n steps.)

Example 1 is ergodic, example 2 is not.

Theorem: If G is an ergodic digraph and T is the transition matrix for G, then as $K \to \infty$, the entries in each column converge to the same number. (This is the probability of being at that vertex).

2 Chapter 8: Matchings

For a bipartite graph with partition U, V a matching of U is a way to choose edges of the graph so that each vertex of U is adjacent to one edge and every vertex of V to at most one edge.
Example: Consider the following graph of children and the books they want to read.

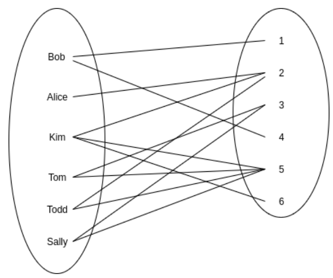


Is there a way to give each kid in the graph one of the books that they want (Edges from kids to books they want) so that every kid gets a book?

The bipartite graph represents the possible values (range) of the elements in the first half (Domain).
Is there a injective function F from U to V such that $F(u) = v$ only if the edge uv was in the graph?

First observation: In order for a matching to exist, we need $|U| \leq |V|$.

Consider the following graph:



In this graph, there are 5 kids that only like 4 books between them so we cannot find a matching.

If $X \subset V(G)$ then $N(X) = \{\text{vertices connected to } x \text{ in } G\}$
Example: $X = \{Alice, Kim\} \to N(X) = \{2, 5, 6\}$
(N for neighbors)

New condition: if $X \subset U$, then to have a matching, we must have that $|X| \leq |N(X)|$.

Hall's Matching Theorem: If G is bipartite with partition U, V then there is a matching from U to V if and only if for every subset $X \subset U, |X| \leq |N(X)|$.

Proof: By strong induction on the number of vertices in U.
Base Case: $|U| = 1$
Supposing that for every subset $X \subset U$, the size $|X| \leq |N(X)|$.
This means the one vertex in U is connected to something in V so there is a matching.
Induction Step: Assume that for some integer k we have proven that for any bipartite graoh $G = (U, V)$ where the size of U is at most k satisfying the conditions of the theorem there exists a matching.
Now suppose that $|U| = k + 1$ and U satisfies the condition that for all $X \subset U, |N(X)| \geq |X|$.
Case 1: The stronger statement that for every $X \subset U$, the inequality $|X| \leq |N(X)|$ holds.
Pick some vertex $u \in U$ and pick a vertex $v \in V$ connected to U.
Consider the graph $G \backslash \{u, v\} = G^1$

Now $U - \{u\}$ has the size K.
For any subset $X \subset U$
$N_{G^1}(X) \subset N_G(X) \backslash \{v\}$
so $|X| \leq |N_{G^1}(X)|$
so we can find a matching in G^1 by induction hypothesis.

proof completed in next class