Notes from 4/12

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Review

Theorem: *A digraph is strong if and only if it has no bridges.*
Proof. −> A strong digraph cannot have a bridge because it is only possible to walk across the bridge in one direction. *Q.E.D*

Theorem: *A nontrivial graph has a strong orientation if and only if it has no bridges.*
Proof. −> A strong digraph cannot have a bridge because it is only possible to walk across the bridge in one direction.

<− Suppose G is a nontrivial graph with no bridges in it. *(Think about a tree—it has bridges. Therefore, this graph G has a lot of cycles.)* Let C be a cycle in G. Orient all the edges around the cycle so that it is possible to get to all the vertices in this cycle. It may be that vertices on C are connected to each other. These edges can be assigned arbitrarily. Let C be a subset of the vertices of the graph that have thus far been assigned directions to their adject edges in such a way that it is possible to get between any two vertices of S. At the beginning, S=C. If S is not the entire graph then there exist vertices of the graph not in S. In particular, we can pick some vertex v that is not in S but is adjacent to a vertex in S. The edge connecting V to S is part of some cycle D.

$$D = v_1, v_2, v_3 \ldots, v_k \in S$$

Let v_i be the first element of D that is part of S. Orient the edges on D with
$V \to V_1, V_1 \to V_2, V_{i-1} \to V_i \in S$. Orient the edge $V_k \to V$.Now, it is possible to walk from V. Any other vertex on D, to any other vertex of S and vice versa. Repeat this until there are no more vertices left not in S. *Q.E.D*

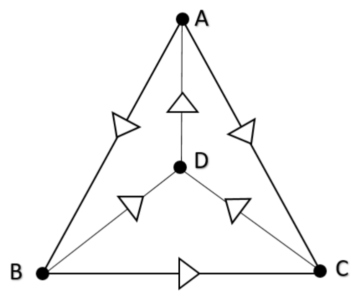
7.2 Tournaments

Def: A <u>Tournament</u> is an orientation of a complete graph. Therefore, a tournament can be defined as a digraph such that for every pair *u,v* of distinct vertices, exactly one of *(u,v)* and *(v,u)* is an arc.. (meaning *u* defeats team *v*).

• Imagine we have n teams that pay in a round robin tournament so that each team plays every other team at last once without ties.
• View this as an orientation of the complete graph K_n in which each edge is directed from the winning team to the losing team.

Example

4 teams - ABCD
A defeats **B**
A defeats **C**
D defeats **A**
B defeats **C**
B defeats **D**
C defeats **D**



Say that two tournaments S, T are isomorphic if they have the same order and there exists a function $\phi T \to S$ such that if $V \to W$ in T then $\phi(v) \to \phi(u)$ in S.

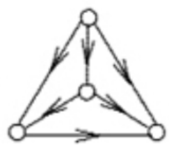
How many tournaments up to isomorphism are there on n teams?

2 teams:

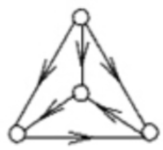
3 teams:

4 teams:

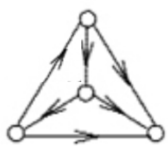
Suppose first that one team wins all off its games.



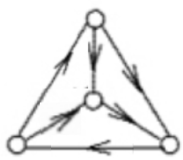
or



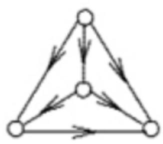
What if some team loses all of its games?



The only remaining possibility for a tournament were no team that either wins or loses all of its games.

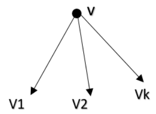


We call a tournament transitive if whenever $u \to v$ and $v \to w$ then $u \to w$.



Theorem: *A tournament is transitive if and only if it does not contain any cycles.*
Proof: In book.

Theorem: *If v is a vertex in a tournament with maximal out degree then* $\vec{d}(u, v) \le 2$ *for every vertex v of T.*
Proof. Let v be a vertex in a tournament T with maximal out degree $\text{od}(v) = K$. Then there are K vertices $v_1, v_2 ... v_k$ that v connects to.



If these are all the vertices in T then $\vec{d}(v, u) \le 1$ for all u.Otherwise there exists vertices $w_1, w_2 ... w$ such that w_1 are connected to v.
Suppose that $\vec{d}$ *(v,w)* > 2. This means that w is connected to all of the vertices v_1, v_2..etc that v was connected to and w is also connected to v.

Thus, $\text{od}(w_i) \ge k + 1$ which contradicts v having maximal out degree. *Q.E.D*

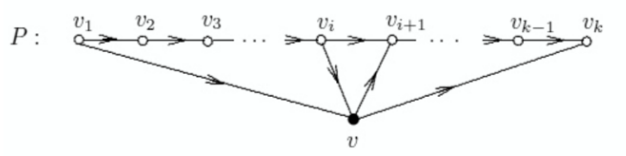
Hamiltonian Path

Def: A <u>Hamiltonian Path</u> in a tournament is a directed path going through all the vertices of the tournament.

Theorem: *Every tournament contains a Hamiltonian path.*
Proof. Let P be a directed path of maximal length in a tournament T.

$$P = \{v_1, v_2 ... v_k\}$$

Suppose P is not Hamiltonian. There exists a vertex w not in P since w cannot go at the beginning or end of P, we must have that $v_1 \to v_w, w \to v_m$.

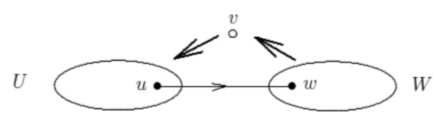


Let v_1 be the first vertex such that $w \to v_1$but $v_{i+1} \to w$. Then we can make a longer directed path,

$$(v_1, v_2 ... v_{i-1}, w, v ... v_k)$$

Q.E.D

Theorem: *If P is a strong tournament (nontrivial) then every vertex $v \in T$ is part of a triangle.*
Proof. Let u be the set of vertices that v is connected to and$w$ the set of vertices connected to v.



Since T is strong, u and v are nonempty and there is at least one edge from $u$to w, say it goes from u to w.

So $v \to u \to w$ is a triangle. *Q.E.D*