Connectivity, Day 1

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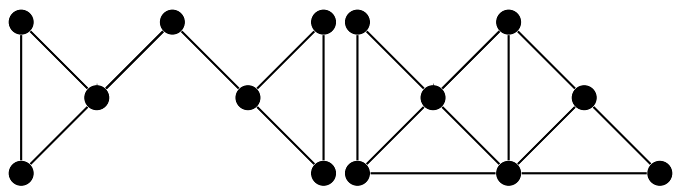
Connectivity Both of the below graphs are connected, however one of the graphs is more connected than the other:



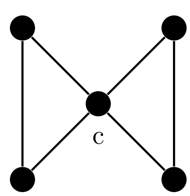
Cut Vertices- A vertex V in a connected graph G, is a cut vertex if G-V is no longer connected.

If a graph G has a bridge e, then one of its ends V is a cut vertex if and only if its degree is at least 2
Proof: If degree (V)=1, then removing the vertex does not disconnect the graph

Other direction (proof by contradiction): Suppose V has a degree of at least 2 and is not a cut vertex. V is connected to one vertex U on the other end of edge e, and at least one other vertex W. Since G-V is still connected, there exists another path from W to U in G-V. This path along with e and the edge from V to W forms a cycle, this contradicts the previous assumption that e is a bridge.

Theorem: If G is connected and has order at least 3, and contains a bridge, it contains a cut vertex.
Proof: Let e be a bridge in G. Since G is connected and has at least 3 vertices, both ends can not have degree 1, so one of the ends is a cut vertex.

Note: graphs can contain cut vertices, without having a bridge.



Observation: In the above graph, the vertex C is a cut vertex, but there is no bridge

Theorem: Suppose that V is a cut vertex in G and U and W are different components of G-V. Then the vertex V must lie on every path from U to W in G.

Proof: Suppose that V is a cut vertex of G. U and W are different components of G-V. Suppose for contradiction that there exists a path from U to W in G that does not contain V. Thus the path still exists in G-V. This contradicts U and W belonging to different components on G-V.

Theorem: V in G is a cut vertex if and only if there exists vertices U and W distinct from V such that V lies on every path from U to W.

Proof: First direction: Suppose V is a cut vertex, then G-V is disconnected. Let U and W be two vertices in different components G-V, proved by the last theorem.

Other direction: Suppose a vertex V exists on every path from U to W in G for some vertices U and W. Then there does not exist a path from U to W in G-V. So G-V is disconnected, thus V is a cut vertex by definition.

Theorem: Let G be a connected graph and U be any vertex in G. Let V be the furthest vertex from U in G. Then vertex V is not a cut vertex

Proof: Suppose G is a connected graph and U is any vertex in G, and V is the furthest vertex from U. Suppose for contradiction that V is not a cut vertex.

Then G-V is disconnected Let W be a vertex in a different component of G-V from U. Then V must lie on every path from U to W by the definition of a cut vertex. However, this would mean the path from U to W is greater than the path from U to V, this contradicts our previous assumption that U and V are as far apart as possible.

Corollary: Every non-trivial connected graph has at least 2 vertices that are not cut vertices.

Proof: Take any vertex U in G. Let V be the vertex furthest from U that is not a cut vertex. Now take the vertex W that is furthest from V, it also can not be a cut vertex. So we have at least 2 uncut vertices.

Notice: This corollary is sharp since any path no matter how long contains only 2 vertices that are not cut vertices. This is because a path always has two end vertices.

non-separable- A graph is called non-separable if it does not contain any cut vertices.

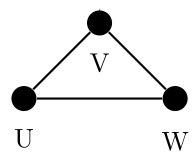
C_n= non-separable
K$_n$= non-separable

P_n= non-separable if n>=3

Theorem: A graph of order at least 3 is non-separable if and only if every two vertices in G lies on a common cycle

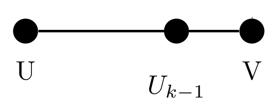
Proof: First direction: Suppose first that any 2 vertices are on a common cycle. Suppose for contradiction that G is not non-separable. This means that G contains a cut vertex, lets call this vertex V.
Then G-V is a disconnected graph, let U and W be in different components of G-V.

By our assumption U and W are on a common cycle...

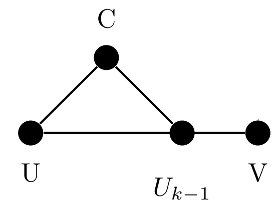


Then there are 2 disjoint paths from U to W, V cannot lie on both of them, so V is not a cut vertex.

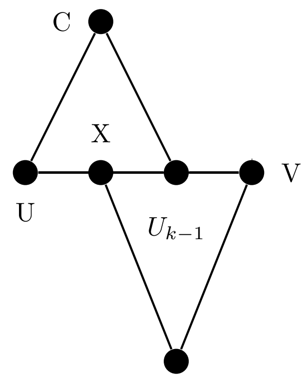
Other direction: Now suppose G is non-separable, but suppose for contradiction that there exists pairs of vertices in G that are not on a common cycle. Lets take two of these vertices not on a common cycle, so that they are as close together as possible.
Call these vertices U and V, say d(V,U)=K. First note that $K \neq 1$ since then the edge is the only way to get from U to V. If this is the case, that would make the edge a bridge, which would lead to the edge connecting U and V to be a bridge. This alludes to a cut vertex, which is not non-separable.
So K>=2
Let U,U_1,U_2,...,U_k=V be a path from U to V. Consider U_{k-1} the last vertex before V on this path.



So since U and U_{k-1} are 1 closes than U and V, they do lie on a common cycle by our assumption that U and V are the closest vertices that do not have a common cycle.



Notice the edge from U_{k-1} to V cannot be a bridge without creating cut vertices, thus it is not a bridge and there must exist some path from V to U that does not use this edge.



Let X be the first vertex along this path that is common with cycle C, containing U, and U_{k-1}. Now the cycle from U to U_{k-1} around C followed by the edge to V, then the path from V to X, and last the other side of C from X to U forms a cycle from U to V. Thus there exists a cycle that contains U and V. This contradicts our assumption that no such cycle existed.