Menger's Theorem

Let G be a graph and u and v be any two vertices in G. Then the size of a minimum uv-seperating set is equal to the maximum number of uv-vertex disjoint paths.

NOTE: If there are k uv-vertex disjoint paths then a minimum uv-separating set must remove at least one vertex from each path.

Thus, it remains to show that the size of a uv-separating set is at most the maximum number of vertex disjoint paths.

Proof. *by induction on the number of edges, m, in the graph*

Base Case ($m = 0$)
There are no edges in this graph so the statement is vacuously true (that is, there is nothing to prove so statement holds).

Induction Step
Suppose the theorem holds true for all graphs of size m and must prove it for graphs of size $m+1$.

Take such a graph (size $m+1$) and let u, v be any two vertices in G.
Suppose the size of a minimum uv separating set is k.
**Goal is to show that there exists K different internally disjoint uv paths.

Case 1

There exists a vertex, w, that is connected to both u and v.
Consider the graph $G - w$. Thus, this graph has at most $m - 1$ edges. Also, this graph has a minimum uv-separating set of size $k - 1$.
Thus, by the induction hypothesis, there exists $k - 1$ internally vertex disjoint uv-paths in the graph $G - w$.
The path uwv is internally disjoint from all of these other ones.
So, we get k-internally vertex disjoint uv paths in G.

Case 2

Let W be a minimum uv-separating set in G. Suppose that at least one vertex in W is not adjacent to u and at least one vertex in W is not adjacent to v.

Create a new graph G_u that consists of all the vertices that lie on the path from u to the vertices of W, along with u and w.

We also throw in one new vertex, v', that is connected to all of the vertices in W.

In G_u, W is a minimum uv'-seperating set, because we suppose at least one vertex in W was not connected to v (directly).
So G_u has size at most m so the induction hypothesis applies and has k internally vertex disjoint uv' paths in G_u.
Repeat the argument, creating a graph, G_v, in the same way as above. So we get k internally

disjoint paths from v to u'.
For each vertex, t_i, in W there exists a path from u to t_i in G_u and a path from t_i to v in G_v. Stitch these two paths together. Thus, we get k internally vertex disjoint paths from u to v.

Case 3

Let W be a minimum uv separating set where either every vertex in W is adjacent to u or every vertex in W is adjacent to v.
Pick a shortest path from u to v in G.

$$(u, x, ..., y, v)$$

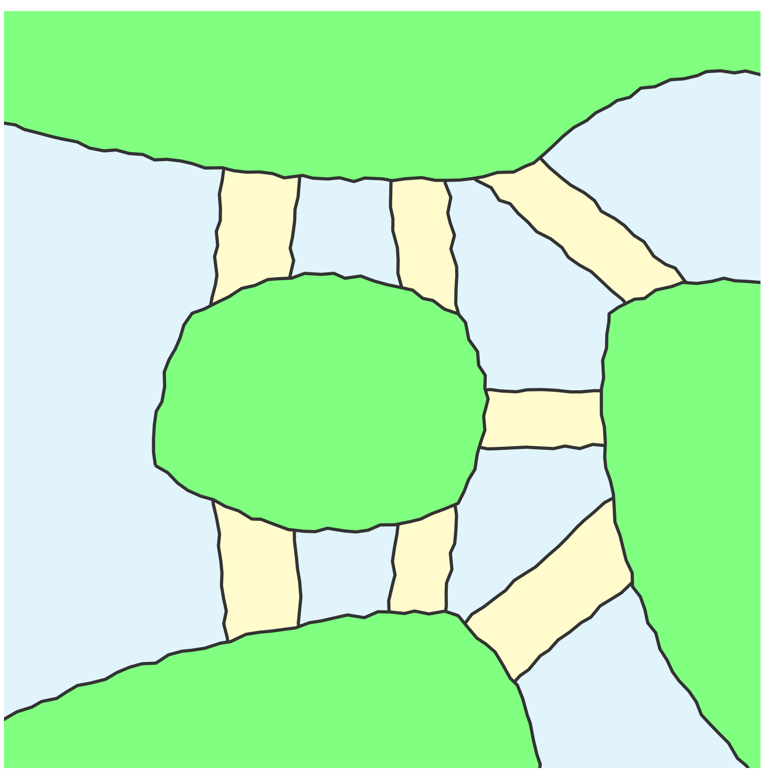
Consider the graph $G - x$. Let W' be a minimum uv-separating set in $G - x$.
Thus, W' has size at least $k - 1$. *(Claim-W' has size k)*

Clearly, $W' \cup \{x\}$ is a uv-separating set in G.
This means: Since x is adjacent to u, that every vertex in W is adjacent to u.

Repeat this argument with y. So again, if W' has size $k - 1$ then every vertex in W is adjacent to v. Thus, the separating set, W' in $G - x$, has size k as well.

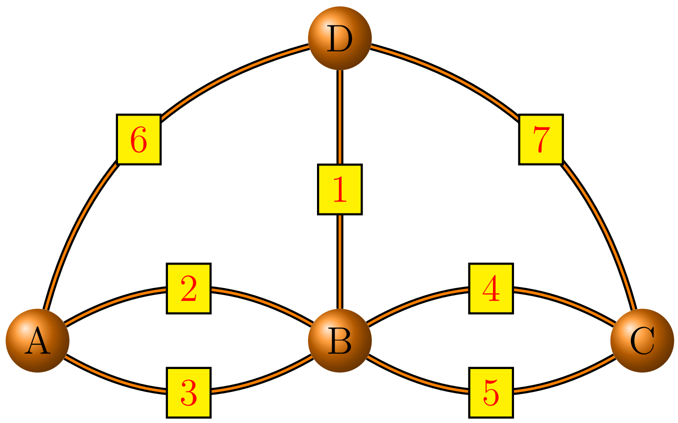
□

KÖNIGSBERG



Question: Is it possible to walk around the city crossing every bridge without crossing the same bridge twice?
Definition: A multigraph is a collection of vertices and edges between vertices where, now, multiple edges are allowed between two vertices.

Example for our question above:



Where A is north side of the city, B is island with 5 bridges connected to it, C is south side of city, and D is other island with only 3 bridges.

**images included were obtained from various blog sites.http://www.texample.net/tikz/examples/bridges-of-konigsberg/ https://tex.stackexchange.com/questions/183882/drawing-k%C3%B6nigsberg-landscape-showing-the-bridges