Math 451 Notes

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κ(G) = minimum number of vertices necessary to remove to make a graph disconnected.

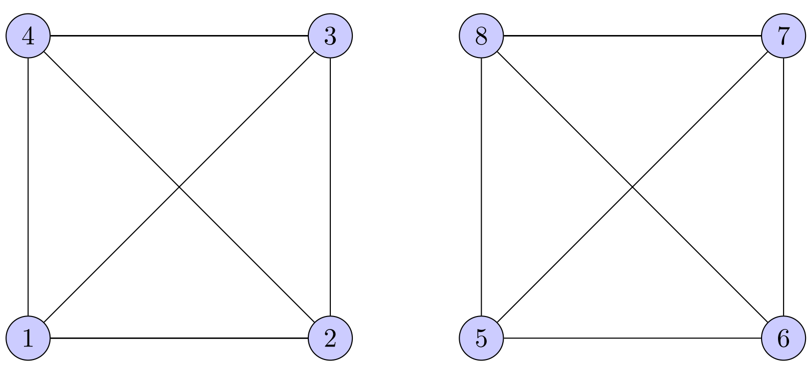
λ(G) = minimum number of of edges required to make a graph disconnected.

Theorem: $\kappa(G) \leq \lambda(G) \leq \delta(G)$

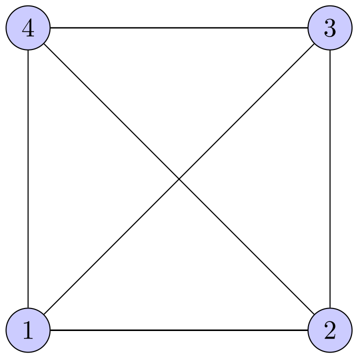
Theorem: If G is a cubic graph then $\kappa(G) = \lambda(G)$. Recall that a cubic graph is a graph where every vertex has degree 3.

Note that in such a graph $\delta(G) \leq 3$ so $0 \leq \kappa(G) \leq 3$.

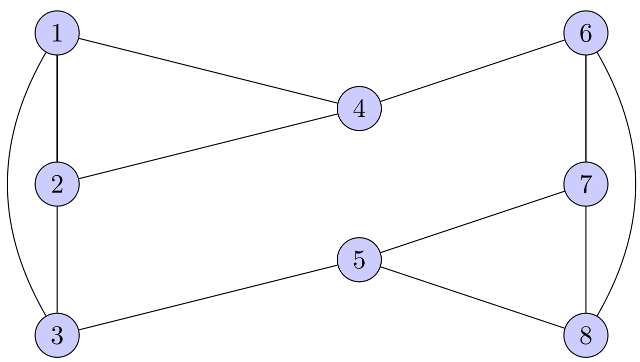
Draw a picture of a cubic graph where $\kappa(G) = 0, 1, 2, 3$.



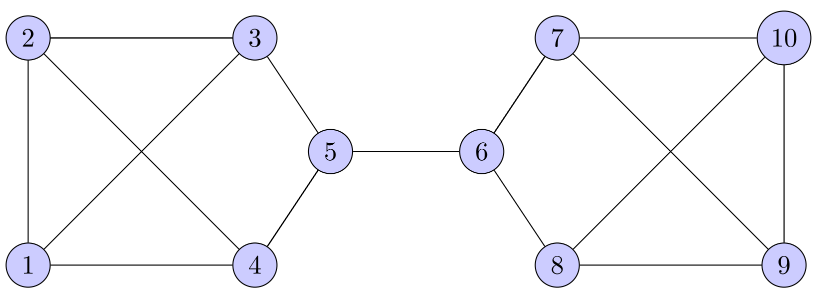
This is a a graph of $\kappa = 0$



This is a a graph of κ(G) = 3



This is a a graph of κ(G) = 2



This is a graph of κ(G) = 1

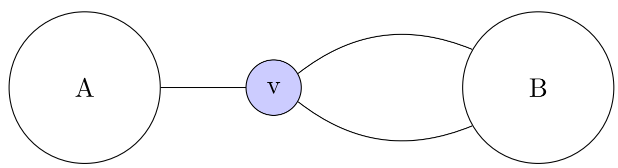
Theorem: If G is a cubic a cubic graph then κ(G) = λ(G).

Proof: Since $\delta(\mathrm{G}) = 3$ then the possibilities for $\kappa(\mathrm{G}) = 0, 1, 2, 3$. This is a proof by cases.

Case 0: Suppose $\kappa(\mathrm{G}) = 0$. This means G is disconnected so no edges need to be removed to disconnect it. $\lambda(\mathrm{G}) = 0 = \kappa(\mathrm{G})$.

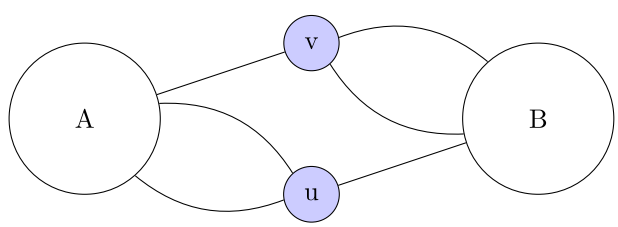
Case 3: Suppose $\kappa(\mathrm{G}) = 0$. Recall that $\kappa(\mathrm{G}) \leq \lambda(\mathrm{G}) \leq \delta(\mathrm{G})$. Since $\delta(\mathrm{G}) = 3$ this means $\lambda(\mathrm{G}) = 3 = \kappa(\mathrm{G})$ as well.

Case 1: Suppose $\kappa(\mathrm{G}) = 1$.



This means that G has a cut vertex v. Removing v creates two disconnected components A and B. So there exists an edge connecting v to A and v to B. Since 3 edges total connect to v one of these components is connected to v by a single edge removing this edge disconnects the graph so $\Lambda(\mathrm{G}) = 1 = \kappa(\mathrm{G})$.

Case 2: Suppose $\kappa(\mathrm{G}) = 2$.



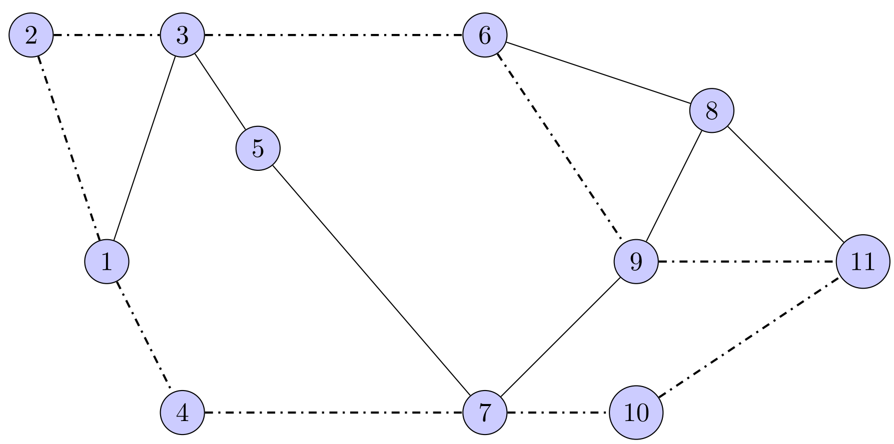
Suppose there exists two vertices u and v that can be removed to disconnect the graph into two components A and B.
If one of these components is connected to both u and v by a single edge then we can remove those edges to disconnect the graph.
Otherwise one vertex is connected to one component by a single edge and the other is vertex is connected to the other component by a single edge and u and

v are not connected.
We can remove these two single edges to disconnect the graph.

Thus λ(G) $\leq$ 2
Since λ(G) $\geq$ κ(G) = 2
We have λ(G) = 2 = κ(G)

Given two points u and v in a graph a uv-separating set S is a collection of vertices such that there is no path from u to v in G-S.

Def: A minimum uv separating set of smallest possible size.
Say a collection of paths from u to v is internally vertex disjoint if no two of them have a vertex in common beside u and v.



The dotted lines are internal vertex disjoint paths.

Vertices 7 and 6 are uv spanning sets.

Menger's Theorem: For any graph G and any two vertices u and v the maximum number of internally vertex disjoint uv paths is equal to the size of a minimum uv separating set.

It is clear that the number of internally disjoint uv paths is at most the size of a minimum uv separating set. This is true because we need to remove one vertex from each of these paths to disconnect u from v.